MACROECONOMIC THEORY
Lecture Notes

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Chapter 1

Introduction to Discrete Time Models

1.1 Basic Robinson Crusoe Economy

It is a common in dynamic macroeconomic models to assume agents live forever. One can think of agents as family dynasties, where those members of the dynasty who are alive today take into account the welfare of all members of the family, including those of generations not yet born. We begin by studying the basic dynamic general equilibrium closed economy model, which assumes that there is only one individual (or a social planner) who makes consumption-savings decisions each period. The social planner chooses $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize lifetime utility, given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1.1}$$

where $u(\cdot)$ is the instantaneous utility function, which is increasing ($u'(\cdot) > 0$), but at a decreasing rate ($u''(\cdot) \leq 0$). The planner values consumption more today than in the future; hence, the discount factor, $0 < \beta < 1$. The planner's choices are constrained by

$$c_t + i_t = y_t, \tag{1.2}$$

$$k_{t+1} = (1-\delta)k_t + i_t, \tag{1.3}$$

$$y_t = f(k_t), \tag{1.4}$$

where c_t, i_t, and y_t are consumption, investment, and output at period t. k_t is the capital stock at the beginning of the period, which depreciates at rate $0 < \delta \leq 1$, and k_{t+1} is the capital stock carried into the next period. Equation (1.2) can either be interpreted as the national income identity (i.e. total output is composed of consumption goods and investment goods) or the aggregate resource constraint (i.e. income is divided between consumption and savings, $s_t = y_t - c_t$, where s_t can only be used to buy investment goods. Equation (1.3) is the law of motion for capital. In period t, a fraction of the capital stock depreciates, δk_t. Thus, the capital stock available in the next period is equal to the fraction that did not depreciate, $(1-\delta)k_t$, plus any investment made in period t. Equation (1.4) is the production function. Output is produced using the capital stock available at the beginning of the period. An increase in capital increases output ($f'(\cdot) > 0$) but at a decreasing rate ($f''(\cdot) \leq 0$). Also

$$\lim_{k \to 0} f'(k) = \infty \qquad \text{and} \qquad \lim_{k \to \infty} f'(k) = 0,$$

which are known as Inada conditions. These equations state that at the origin, there are infinite output gains to increasing capital, but the gains decline and eventually approach 0. We can combine

(1.2)-(1.4) to obtain

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t. \tag{1.5}$$

The goal of the planner is to make consumption-savings decisions in every period that maximize (1.1) subject to (1.5).

1.1.1 Euler Equation

We can substitute for c_t in (1.1) using (1.5) to reduce the problem to an unconstrained maximization problem. In this case, the planner's problem is to choose $\{k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}),$$

which is equivalent to differentiating

$$\cdots + \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta^{t+1} u(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}) + \cdots$$

with respect to k_{t+1}. Thus, we obtain the following first order condition

$$-\beta^t u'(c_t) + \beta^{t+1} u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta] = 0,$$

which, after simplification, becomes

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]. \tag{1.6}$$

This equation is known as an *Euler equation*, which is the fundamental dynamic equation in intertemporal optimization problems that include dynamic constraints. It relates the marginal utility of consumption at time t to the discounted marginal utility benefit of postponing consumption for one period. More specifically, it states that marginal cost of forgoing consumption today must equal to discounted marginal benefit of investing in capital for one period.

The optimal consumption-savings decision must also satisfy

$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0, \tag{1.7}$$

which is known as the *transversality condition*. To understand the role of this condition in intertemporal optimization, consider the implication of having a finite capital stock at time T. If consumed, this would yield discounted utility equal to $\beta^T u'(c_T)k_T$. If the time horizon was also T, then it would not be optimal to have any capital left in period T, since it should have been consumed instead. Hence, as $t \to \infty$, the transversality condition provides as extra optimality condition for intertemporal infinite-horizon problems.

1.1.2 Dynamic Programming

Although we stated the problem in section 1.1 as choosing infinite sequences for consumption and capital, the problem the planner faces at time $t = 0$ can be viewed more simply as choosing today's consumption and tomorrow's beginning of period capital.[1] The value function, $V(k_0)$, is given by

$$V(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{1.8}$$

[1] The presentation in this section is kept simple, with the hope of communicating the main ideas quickly and enabling the reader to use these techniques to solve problems. For a more thorough presentation see Stoky et al. (1989).

It represents the maximum value of the objective function, given an initial level of capital at time $t = 0$, k_0. $V(k_1)$ is the maximum value of utility that is possible when capital at time $t = 1$ is k_1, and $\beta V(k_1)$ discounts this value to time $t = 0$. Thus, we can rewrite (1.8) as

$$\begin{aligned} V(k_0) &= \max_{c_0,k_1} \left[u(c_0) + \max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t u(c_t) \right] \\ &= \max_{c_0,k_1} \; [u(c_0) + \beta V(k_1)], \end{aligned}$$

which is known as Bellman's functional equation. The study of dynamic optimization problems through the analysis of such functional equations is called *dynamic programming*. When we look at the problem in this recursive way, the time subscripts are unnecessary, since the date is irrelevant for the optimal solution. After substituting for c_t using (1.5), Bellman's equation, conditional on the state at time t is

$$V(k_t) = \max_{k_{t+1}} [u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta V(k_{t+1})]. \tag{1.9}$$

The first order condition is given by

$$-u'(c_t) + \beta V'_{t+1}(k_{t+1}) = 0. \tag{1.10}$$

To illustrate the *Envelope Condition*, postulate a law of motion for capital given by $k_{t+1} = h(k_t)$, which intuitively asserts that tomorrow's capital stock is a function of today's stock. Next, substitute this "optimal investment plan" into the initial problem, (1.9), to obtain

$$V(k_t) = u(f(k_t) + (1-\delta)k_t - h(k_t)) + \beta V(h(k_t)).$$

Optimizing with respect to the state variable, k_t, yields

$$\begin{aligned} & V'(k_t) = u'(c_t)[f'(k_t) + 1 - \delta - h'(k_t)] + \beta V'(k_{t+1})h'(k_t) \\ \rightarrow \quad & V'(k_t) = u'(c_t)[f'(k_t) + 1 - \delta] + [-u'(c_t) + \beta V'(k_{t+1})]h'(k_t). \end{aligned}$$

The expression multiplying $h'(k_t)$ is zero according to (1.10). Thus, the expression simplifies to

$$V'(k_t) = u'(c_t)[f'(k_t) + 1 - \delta]. \tag{1.11}$$

In short, the Envelope Condition allows us to differentiate (1.9) directly with respect to the state variable k_t, ignoring its effect on k_{t+1} to get the exact same result.

If we advance equation (1.11) forward one period, we can use the result to substitute for $V'(k_{t+1})$ in (1.10) to obtain

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta], \tag{1.12}$$

which is the same as the Euler equation given in (1.6). Note that we could have alternatively used (1.5) to substitute for k_{t+1} in (1.8) and applied similar steps to obtain the same Euler equation.

1.1.3 Lagrange Method

We could also solve the constrained maximization problem using the Lagrange method. The Lagrangian is given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t [f(k_t) + (1-\delta)k_t - k_{t+1} - c_t]\},$$

where the multiplier on the constraint is $\beta^t\lambda_t$. There are two choice variables, c_t and k_{t+1}. The first-order conditions with respect to these variables are

$$\beta^t[u'(c_t) - \lambda_t] = 0, \tag{1.13}$$
$$-\beta^t\lambda_t + \beta^{t+1}\lambda_{t+1}[f'(k_{t+1}) + 1 - \delta] = 0. \tag{1.14}$$

The Lagrage multiplier is easily obtained from (1.13). Substituting for λ_t and λ_{t+1} in (1.14) yields

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta],$$

which, once again, is the same Euler equation given in (1.6).

1.1.4 Intuitive Derivation of the Euler Equation

If we reduce c_t by a small amount, dc_t, how much larger must c_{t+1} be to fully compensate, leaving utility across the two periods unchanged? Define total utility in any two consecutive periods as

$$V_t = u(c_t) + \beta u(c_{t+1}).$$

Keeping total utility constant, the total derivative is

$$0 = dV_t = u'(c_t)dc_t + \beta u'(c_{t+1})dc_{t+1}. \tag{1.15}$$

The loss in utility is $u'(c_t)dc_t$. In order for V_t to remain constant, this loss must be compensated by the discounted gain in utility $\beta u'(c_{t+1})dc_{t+1}$.

The resource constraint, (1.5), must also be satisfied in every period. Totally differentiating the resource constraints in periods t and $t+1$ implies

$$dc_t + dk_{t+1} = f'(k_t)dk_t + (1-\delta)dk_t,$$
$$dc_{t+1} + dk_{t+2} = f'(k_{t+1})dk_{t+1} + (1-\delta)dk_{t+1}.$$

Since k_t is given and beyond period $t+1$ we are constraining the capital stock to be unchanged, $dk_t = dk_{t+2} = 0$. Hence

$$dc_t + dk_{t+1} = 0,$$
$$dc_{t+1} = f'(k_{t+1})dk_{t+1} + (1-\delta)dk_{t+1},$$

which can be combined by eliminating dk_{t+1} to obtain

$$dc_{t+1} = -[f'(k_{t+1}) + 1 - \delta]dc_t. \tag{1.16}$$

This is an indifference curve that trades consumption tomorrow for consumption today. The output no longer consumed in period t is invested and increases output in period $t+1$ by $-f'(k_{t+1})dc_t$. This amount, in addition to the undepreciated increase in the capital stock $-(1-\delta)dc_t = (1-\delta)dk_{t+1}$, gives the total increase in consumption in period $t+1$. Plugging this value in for dc_{t+1} in (1.15) implies

$$u'(c_t)dc_t = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]dc_t.$$

Cancelling out dc_t yields the same Euler equation given in (1.6).

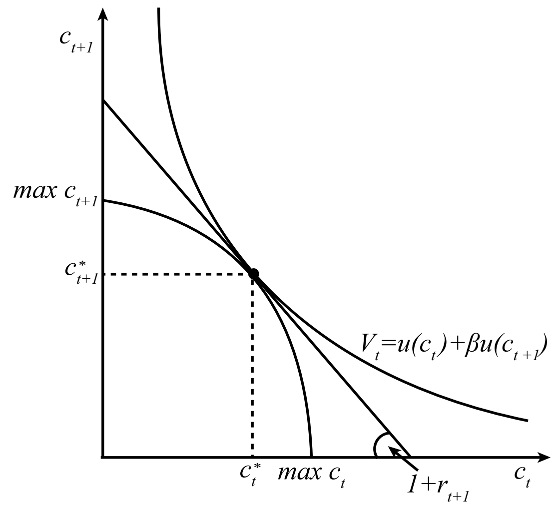

Figure 1.1: Graphical solution to the basic dynamic general equilibrium model

1.1.5 Graphical Solution

The production possibility frontier is associated with a production function with more than one type of output. It measures the maximum combination of each type of output that can be produced with a fixed amount of factors. The intertemporal production possibility frontier (IPPF) is associated with output at different points in time and is derived from the resource constraint, (1.5). Combine the constraints at periods $t+1$ and $t+2$ to eliminate k_{t+1} and obtain the IPPF, given by,

$$\begin{aligned} c_{t+1} &= f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1} \\ &= f(f(k_t) - c_t + (1-\delta)k_t) - k_{t+2} + (1-\delta)[f(k_t) - c_t + (1-\delta)k_t], \end{aligned} \tag{1.17}$$

which is a concave relation between c_t and c_{t+1}. The slope of the IPPF is

$$\frac{\partial c_{t+1}}{\partial c_t} = -[f'(k_{t+1}) + 1 - \delta],$$

which is concave given that $\partial^2 c_{t+1}/\partial^2 c_t = f''(k_{t+1}) < 0$.

The solution to the two period problem is represented in figure 1.1. The upper curved line is the indifference curve, given in (1.16). The lower curved line is the IPPF, which touches the indifference curve at the point of tangency with the budget constraint. Hence, the solution must satisfy

$$-\left.\frac{dc_{t+1}}{dc_t}\right|_{V_{\text{const.}}} = f'(k_{t+1}) + 1 - \delta = 1 + r_{t+1} = -\left.\frac{\partial c_{t+1}}{\partial c_t}\right|_{\text{IPPF}},$$

where the net marginal product, $f'(k_{t+1}) - \delta = r_{t+1}$ represents the implied real rate of return on capital. An increase in r_{t+1} makes the resource constraint steeper, which increases V_t, c_t, and c_{t+1}.

1.1.6 Stability and Saddlepath Dynamics

A useful graphical tool for studying two-dimensional nonlinear dynamic systems is a phase diagram. To construct the phase diagram presented in figure 1.2, consider the two equations that describe the optimal solution at each point in time—the Euler equation and the resource constraint, which are reproduced below:

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{1.18}$$
$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]. \tag{1.19}$$

We first consider the long-run equilibrium properties. The long-run equilibrium is a static solution, implying that in the absence of shocks to the macroeconomic system, consumption and the capital stock will be constant through time. Thus, $c_t = c^*$, $k_t = k^*$, $\Delta c_t = 0$, and $\Delta k_t = 0$ for all t. In static equilibrium the Euler equation can be written as

$$f'(k^*) = \beta^{-1} + \delta - 1 = \delta + \theta,$$

where $\theta \equiv \beta^{-1} - 1$.This condition shows that the steady state level of capital is independent of consumption. We depict this on the phase diagram in (k, c)-space with a vertical line. To see what happens to consumption as $k \lesseqgtr k^*$, note that the Euler equation, (1.19), implies

$$\begin{aligned} c_{t+1} \geq c_t &\iff \beta u'(c_{t+1}) \leq \beta u'(c_t) \\ &\iff \frac{u'(c_t)}{f'(k_{t+1}) + 1 - \delta} \leq \beta u'(c_t) \\ &\iff 1 \leq \beta(f'(k_{t+1}) + 1 - \delta) \\ &\iff \delta + \theta \leq f'(k_{t+1}) \\ &\iff k_{t+1} \leq k^*. \end{aligned}$$

Thus, whenever $k \lesseqgtr k^*$, $\Delta c \gtreqless 0$, which is represented by vertical arrows. From the budget constraint, it is easy to see that $\Delta k = 0$ implies

$$c = f(k) - \delta k,$$

which we can depict on the phase diagram with a hump-shaped curve that peaks at $\bar{k} = \delta > k^*$. To see what happens to capital above and blow this line, note that the budget constraint, (1.18), implies

$$\begin{aligned} k_{t+1} \geq k_t &\iff f(k_t) + (1-\delta)k_t - c_t \geq k_t \\ &\iff c_t \leq f(k_t) - \delta k_t. \end{aligned}$$

Thus, whenever $c \lesseqgtr f(k) - \delta k$, $\Delta k \gtreqless 0$, which is represented by horizontal arrows.

Figure 1.2 shows that there is a unique level of capital where the two lines intersect (point B). Thus, a steady state (k^*, c^*) that satisfies the equilibrium conditions must exist and is unique. Note that the origin $(0, 0)$ is also a steady state, since an economy that begins with zero capital remains at $(0, 0)$. However, this steady state violates the Euler equation, since $\lim_{k\to 0} f'(k) = \infty$. Thus, trajectories that converge to the vertical axis are not equilibria. Likewise, trajectories that converge to the intersection of the $\Delta k = 0$ schedule and the horizontal axis do not reach an equilibrium since the transversality condition, given in (1.7) is violated. To see this note that (1.19) implies

$$\frac{u'(c_{t+1})}{u'(c_t)} - 1 = \frac{1}{\beta(f'(k_{t+1}) + 1 - \delta)} - 1 > \frac{1}{\beta} - 1.$$

The inequality follows from that fact that at this point, $k > \bar{k}$, which implies $f'(k) < \delta$. In other words, the rate of growth of $u'(c_t)$ is larger than the rate of decline of the discount factor, $1/\beta - 1$. Since k is constant, this implies the transversality condition is violated.

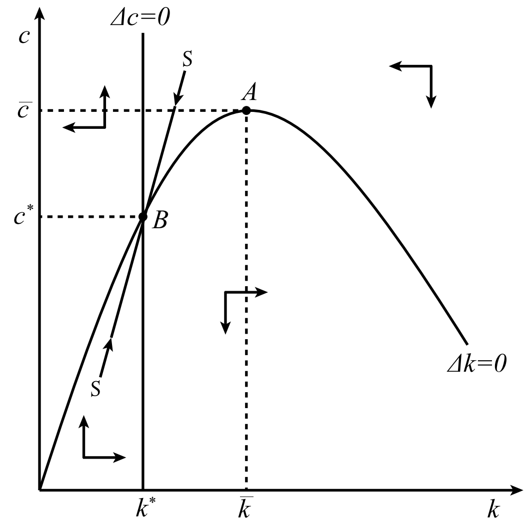

Figure 1.2: Phase diagram to the basic dynamic general equilibrium model

The SS line through point B is known as the saddlepath. Only trajectories that converge to the intersection of (k^*, c^*) satisfy the equilibrium conditions. Given the direction of the arrows, it seems likely that the steady state, given by (k^*, c^*), is reachable through trajectories that follow SS. In the northeast quadrant, consumption is excessive and the capital stock is so large that the marginal product of capital is less than $\delta + \theta$. This is not sustainable and therefore consumption and the capital stock must decrease. The opposite is true in the southwest region. The other two regions, as discussed above, are not stable.

Equations (1.18) and (1.19) represent a first-order nonlinear difference equation in the vector (k_t, c_t). To better understand the stability properties of the model, level linearize the system of equations by taking a first-order Taylor series approximation around the steady state to obtain

$$u''(c^*)(c_t - c^*) = \beta u''(c^*)(f'(k^*) + 1 - \delta)(c_{t+1} - c^*) + \beta u'(c^*) f''(k^*)(k_{t+1} - k^*)$$
$$c_t - c^* = f'(k^*)(k_t - k^*) + (1 - \delta)(k_t - k^*) - (k_{t+1} - k^*).$$

Given that percent changes are a good approximation for log deviations, we can represent this system in log-linear form as

$$\hat{c}_t = \hat{c}_{t+1} - \beta\sigma^{-1} f''(k^*) k^* \hat{k}_{t+1}$$
$$c^* \hat{c}_t / k^* = \beta^{-1} \hat{k}_t - \hat{k}_{t+1},$$

where $\hat{x}_t = \ln x_t - \ln x^* \approx (x_t - x^*)/x^*$ and $\sigma \equiv -u''(c)c/u'(c) \geq 0$ is the coefficient of relative risk aversion. In derivaing these equations, we made use of the fact that in stationary equilibrium $1 = \beta[f'(k^*) + 1 - \delta]$. The system of equations can be rewritten in matrix form as

$$\begin{bmatrix} -\beta\sigma^{-1} f''(k^*) k^* & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \beta^{-1} & -c^*/k^* \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}.$$

Defining $\chi = -\beta f''(k^*) k^* / \sigma > 0$, we obtain

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \beta^{-1} & -c^*/k^* \\ -\chi\beta^{-1} & 1 + \chi c^*/k^* \end{bmatrix}}_{A} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}.$$

The eigenvalues of the matrix, A, are the solutions to the characteristic equation, given by,

$$p(\lambda) = \lambda^2 - (1 + \beta^{-1} + \chi c^*/k^*)\lambda + \beta^{-1}.$$

Thus, the trace, T, and determinant, D, of the matrix, A, are given by

$$T = 1 + \beta^{-1} + \chi c^*/k^* > 1 + \beta^{-1} > 2 \qquad \text{and} \qquad D = \beta^{-1} > 1.$$

Hence, the discriminant, $T^2 - 4D > (1 + \beta^{-1})^2 - 4\beta^{-1} = (1 - \beta^{-1})^2 > 0$, and the characteristic polynominal has two positive real roots, whose sum exceeds 2 and whose product exceeds 1. Since $p(1) = -\chi c^*/k^* < 0$, the two real eigenvalues lie on either side of unity, which means that $0 < \lambda_1 < 1 < \lambda_2$ and (k^*, c^*) is a saddle.

If we define $\mathbf{z}_t = [\hat{k}_t \;\; \hat{c}_t]^T$, then our difference equation is $\mathbf{z}_t = A\mathbf{z}_{t-1}$. Let $D = \text{diag}(\lambda_1, \lambda_2)$ and P be the corresponding projection matrix. Then if we define $Z = P^{-1}\mathbf{z}$, we obtain $Z_t = DZ_{t-1}$. Hence $Z_{1,t} = \lambda_1 Z_{1,t-1}$ and $Z_{2,t} = \lambda_2 Z_{2,t-1}$. The general solutions are $Z_{1,t} = a_1\lambda_1^t$ and $Z_{2,t} = a_2\lambda_2^t$ for some constants a_1 and a_2. To recover the solutions for $\hat{k}_t$ and $\hat{c}_t$, apply the projection matrix to Z_t to obtain

$$\begin{aligned}\mathbf{z}_{1,t} &= P_{11}Z_{1,t} + P_{12}Z_{2,t} = P_{11}a_1\lambda_1^t + P_{12}a_2\lambda_2^t \\ \mathbf{z}_{2,t} &= P_{21}Z_{1,t} + P_{22}Z_{2,t} = P_{21}a_1\lambda_1^t + P_{22}a_2\lambda_2^t,\end{aligned}$$

where P_{ij} $(i, j \in \{1, 2\})$ are the elements of the projection matrix. Since k_0 is given, $\mathbf{z}_{1,0} = \hat{k}_0 = (k_0 - k^*)/k^*$ is also given. Hence

$$\hat{k}_0 = P_{11}a_1 + P_{12}a_2 \tag{1.20}$$

We also know that the optimal paths for c_t and k_t converge to the steady state. Thus, $\lim_{t\to\infty} \mathbf{z}_{1t} = \lim_{t\to\infty} \mathbf{z}_{2t} = 0$. This implies that $a_2 = 0$, since $\lambda_2 > 1$. From (1.20), $a_1 = \hat{k}_0/P_{11}$, and the solutions for $\hat{k}_t$ and $\hat{c}_t$ are

$$\begin{aligned}\hat{k}_t &= \hat{k}_0\lambda_1^t \\ \hat{c}_t &= P_{21}\hat{k}_0\lambda_1^t/P_{11} = P_{21}\hat{k}_t/P_{11},\end{aligned}$$

which is a the unique stable solution that converges to the stationary equilibrium.

1.2 Extensions to the Basic Robinson Crusoe Economy

1.2.1 Endogenous Labor Supply

In the basic model, labor is inelastically supplied. That is, since agents did not derive utility from leisure, the optimal choice of labor was to spend all available time working. To extend the basic model, we assume leisure provides utility, so that the planner makes an endogenous labor supply decision about how much time to spend working and consuming leisure. We assume that the total available time is 1. Thus, $n_t + \ell_t = 1$, where n_t is hours worked and ℓ_t is leisure. The instantaneous utility function is $u(c_t, \ell_t)$, where $u_c > 0$, $u_\ell > 0$, $u_{cc} \leq 0$, and $u_{\ell\ell} \leq 0$. This says that there is positive, but diminishing marginal utility to both consumption and leisure. For convenience, we assume $u_{c\ell} = 0$, which rules out substitution between consumption and leisure. We also assume labor is a second factor of production. Thus, the production function becomes $f(c_t, n_t)$, with $f_k > 0$, $f_n > 0$, $f_{kk} \leq 0$, $f_{nn} \leq 0$, $f_{kn} \geq 0$, and $\lim_{k\to\infty} f_k = 0$, $\lim_{n\to\infty} f_n = 0$, $\lim_{k\to 0} f_k = \infty$, and $\lim_{n\to 0} f_n = 0$, which are the Inada conditions.

The problem is as follows. A representative planner chooses sequences $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize lifetime utility given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to the resource constraint,

$$c_t + k_{t+1} - (1 - \delta)k_t = f(k_t, n_t),$$

where we have substituted the time constraint, $n_t + \ell_t = 1$, into the instantaneous utility function.

The Lagrangian is given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{u(c_t, 1 - n_t) + \lambda_t (f(k_t, n_t) - k_{t+1} + (1 - \delta)k_t - c_t)\},$$

where λ_t is the Lagrange multiplier on the resource constraint. The first order conditions are given by

$$\begin{aligned} c_t: &\quad u_{c,t} = \lambda_t \\ n_t: &\quad u_{\ell,t} = \lambda_t f_{n,t} \\ k_{t+1}: &\quad \lambda_t = \beta \lambda_{t+1} [f_{k,t+1} + 1 - \delta]. \end{aligned}$$

The first order conditions for c_t and k_{t+1} yield the same Euler equation as the model where labor is inelastically supplied, and is given by,

$$u_{c,t} = \beta u_{c,t+1} [f_{k,t+1} + 1 - \delta].$$

The first order condition for labor implies

$$u_{\ell_t} = u_{c,t} f_{n,t}.$$

This equation says that the loss in utility from giving up $dl_t = -dn_t < 0$ units of leisure is compensated by an increase in utility due to producing extra output equal to $-u_{c,t} f_{n,t} d\ell_t > 0$.

To summarize, we have found that when we allow the planner to choose how much to work, the solutions for consumption and capital are virtually unchanged from those of the basic model.

1.2.2 Investment Adjustment Costs

The basic model includes investment, but thus far we have assumed that there are no costs to installing new capital. Suppose there is a permanent change in the long-run equilibrium level of capital. Although capital takes time to adjust to its new steady state level, investment in the basic model adjusts instantaneously to the level that is optimal each period. In practice, however, it is usually optimal to adjust investment more slowly, due to installation costs

To illustrate, suppose that new investment imposes an additional resource cost equal to $\phi i_t/(2k_t)$ for each unit of investment, where $\phi \geq 0$. In this case, the cost of a unit of investment depends on how large it is in relation to the size of the existing capital stock. The planner's decisions are now constrained by

$$f(k_t) = c_t + \left(1 + \frac{\phi}{2} \frac{i_t}{k_t}\right) i_t \tag{1.21}$$

$$k_{t+1} = (1 - \delta)k_t + i_t. \tag{1.22}$$

The Lagrangian is given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left(f(k_t) - i_t - \frac{\phi}{2}\frac{i_t^2}{k_t} - c_t \right) + \mu_t(i_t + (1-\delta)k_t - k_{t+1}) \right\},$$

where λ_t is the Lagrange multiplier on (1.21) and μ_t is the Lagrange multiplier on (1.22). The first order conditions are given by

$$\begin{aligned} c_t: &\quad u_{c,t} = \lambda_t \\ i_t: &\quad \mu_t = \lambda_t\left(1 + \phi\frac{i_t}{k_t}\right) \\ k_{t+1}: &\quad \mu_t = \beta\lambda_{t+1}\left[f_{k,t+1} + \frac{\phi}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2\right] + \beta\mu_{t+1}(1-\delta). \end{aligned}$$

The first order condition for investment implies

$$q_t = 1 + \phi(i_t/k_t) \qquad \to \qquad i_t = (q_t - 1)k_t/\phi,$$

where the ratio of the Lagrange multipliers $q_t = \mu_t/\lambda_t \geq 1$ is known as Tobin's q. There exists positive investment if $q_t > 1$. λ_t is the marginal utility value, in terms of net output, of an additional unit of k. μ_t measures the marginal utility value of 1 unit of investment. Hence, q_t represents the benefit from investment per unit of benefit from capital in terms of units of output. Or, q_t is the market value of 1 unit of investment relative to its replacement costs.

1.3 Competitive Economy

In the Robinson Crusoe economy, one person made consumption and production decisions for the whole economy. In a competitive economy, consumers rent capital to firms and sell labor. In this section, we assume there is a continuum of identical agents of a unit mass and that all agents can provide up to one unit of labor to the market. All agents are the same so that we can take the behavior of one agent as that of the whole economy since we simply integrate from 0 to 1 over identical agents.

1.3.1 Consumer's Problem

Individual i chooses sequences $\{c_t^i, n_t^i, k_{t+1}^i\}$ to maximize lifetime utility, given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i, n_t^i)$$

subject to

$$c_t^i + k_{t+1}^i = w_t n_t^i + r_t k_t^i + (1-\delta)k_t^i, \tag{1.23}$$

where w is the wage rate and r is the rental rate on capital.

The Lagrangian is given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{u(c_t^i, n_t^i) + \lambda_t(w_t n_t^i + r_t k_t^i + (1-\delta)k_t^i - c_t^i - k_{t+1}^i)\}$$

where λ_t is the Lagrange multiplier on (1.23). The first order conditions are given by

$$\begin{aligned} c_t^i: &\quad u_{c,t} = \lambda_t \\ n_t: &\quad u_{\ell,t} = \lambda_t w_t \\ k_{t+1}: &\quad \lambda_t = \beta\lambda_{t+1}[r_{t+1} + 1 - \delta]. \end{aligned}$$

After combining these results, we obtain

$$\begin{aligned} u_{\ell,t} &= u_{c,t} w_t \\ u_{c,t} &= \beta u_{c,t+1}[r_{t+1} + 1 - \delta] \end{aligned}$$

The first equation says that the loss in utility from giving up $dl_t = dn_t < 0$ units of leisure is compensated by an increase in utility due to earning w_t. The second equation says that the marginal cost of foregoing consumption today in favor of investing in capital is equal to the discounted utility value of that investment tomorrow. The net return on investment equals $1 + r_{t+1} - \delta$.

1.3.2 Firm's Problem

The firm maximizes profit each period by choosing $\{k_t, n_t\}$ to maximize

$$f(k_t, n_t) - w_t n_t - r_t k_t.$$

The first order conditions are

$$\begin{aligned} f_k(k_t, n_t) &= r_t \\ f_n(k_t, n_t) &= w_t. \end{aligned}$$

These equations show that this is a competitive firm as each of the factor prices is equal to its marginal product.

1.3.3 Competitive Equilibrium

The competitive equilibrium system is composed of the consumer's and firm's optimality conditions,

$$\begin{aligned} u_{\ell,t} &= u_{c,t} f_{n,t} \\ u_{c,t} &= \beta u_{c,t+1}[f_{k,t+1} + 1 - \delta], \end{aligned}$$

the budget constraint, (1.23), and the aggregation rules,

$$n_t = \int_0^1 n_t^i di \qquad \text{and} \qquad k_t = \int_0^1 k_t^i di.$$

When all the unit mass of individuals are identical, as in the case, the aggregation rules simplify to $n_t = n_t^i$ and $k_t = k_t^i$.

The production function is homogeneous of degree one (constant returns to scale) and under conditions of perfect competition with free entry, firms to do not make any profits. Hence

$$f_{n,t} n_t + f_{k,t} k_t = f(k_t, n_t),$$

and the budget constraint can be rewritten as

$$c_t + k_{t+1} = f(k_t, n_t) + (1-\delta)k_t,$$

where aggregate consumption is $c_t = \int_0^1 c_t^i di$.

Notice that the conditions for the equilibrium in the competitive economy turn out to be an aggregate version of the same conditions for the Robinson Crusoe economy. The equilibrium that we found to the Robinson Crusoe economy is Pareto optimal. It is the result of the social planner finding consumption and production points that maximize the utility of the single individual in the economy, given his technology constraints. The first fundamental welfare theorem tells us that any competitive equilibrium is necessarily Pareto optimal, so that the equilibrium found using a decentralized economy with factor and goods markets is also Pareto optimal. The second welfare theorem tells us that, since the production technologies and preferences are the same in the two economies, then with the right initial wealth conditions, the competitive equilibrium can achieve an equilibrium that is identical to the social planner economy.

It is the second fundamental welfare theorem that permits us to use Robinson Crusoe economy to mimic a competitive economy. Since the second fundamental theorem is carefully worded, it should be clear that the solution to the social planner's problem will not always give the appropriate results. If the economy is not perfectly competitive, if part of the economy has some monopoly power or if there are some internal or external restrictions that prevent some agents from behaving perfectly competitive, then the economy found in the decentralized economy will not necessarily be achievable in the Robinson Crusoe economy.

1.4 Solution Methods

There are several different ways to solve nonlinear dynamic models:

1. Method of undetermined coefficients (guess and verify method): Guess a functional form for the value function and then verify that this functional form satisfies Bellman's functional equation.

2. Value function iteration (Bellman Iteration)

3. Euler equation iteration

4. Howard's Improvement Algorithm: guess a functional form for the policy function and plug it into the Euler equation.

Some solution methods will be easier to apply to certain models. Below is set of examples that are designed to help you develop the basic techniques involved in each of the above solution methods. Note, however, that these are very stylistic examples. Most models can not be solved analytically and will be require numerical techniques.

1.4.1 Method of Undetermined Coefficients

Example 1: Cake-Eating Problem

We begin with a very simple dynamic optimization problem. Suppose that you have a cake of size w_t. At each point in time, $t = 1, 2, 3, \ldots$ you eat some of the cake and save the remainder. The planner chooses $\{c_t, w_{t+1}\}$ to maximize lifetime utility given by (1.1). Assume that the cake does not depreciate or grow. Hence, the planner's choices are constrained by

$$c_t + w_{t+1} = w_t, \tag{1.24}$$

which governs the size of the cake. Given an initial size of the cake, w_0, the Bellman equation is

$$V(w_t) = \max_{c_t, w_{t+1}} \{u(c_t) + \beta V(w_{t+1})\}.$$

After substituting for c_t using (1.24), the the problem reduces to

$$V(w_t) = \max_{w_{t+1}} \{u(w_t - w_{t+1}) + \beta V(w_{t+1})\}. \tag{1.25}$$

In general, actually finding closed form solutions for the value function and the resulting policy functions is not possible. In those cases, we try to characterize certain properties of the solution and, for some exercises, we solve these problems numerically. However, there are some versions of the problem we can solve for analytically. Suppose $u(c) = \ln(c)$ and guess that the value function has function form $V(w) = A + B \ln w$. With this guess we have reduced the dimensionality of the unknown function, $V(w)$ to two parameters, A and B. The question is whether we can find values for A and B such that $V(w)$ will satisfy the functional equation. Taking this guess as given and using the log preferences, the functional equation becomes

$$A + B \ln w_t = \max_{w_{t+1}} \{\ln(w_t - w_{t+1}) + \beta(A + B \ln w_{t+1})\}. \tag{1.26}$$

The first order condition implies

$$-\frac{1}{w_t - w_{t+1}} + \frac{\beta B}{w_{t+1}} = 0.$$

Thus, after simplification, we obtain

$$w_{t+1} = \frac{\beta B w_t}{1 + \beta B}. \tag{1.27}$$

Plugging the result of (1.27) into the value function (1.26), we obtain

$$A + B \ln w_t = \ln\left(\frac{w_t}{1 + \beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B w_t}{1 + \beta B}\right).$$

Equating coefficients, we find $B = 1 + \beta B$, which implies $B = 1/(1 - \beta)$. Hence,

$$\begin{aligned} A &= \ln\left(\frac{1}{1 + \beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B}{1 + \beta B}\right) \\ &= \frac{1}{1 - \beta}\left[\ln(1 - \beta) + \frac{\beta}{1 - \beta} \ln \beta\right]. \end{aligned}$$

Plugging the value of B into (1.27) and utilizing the budget constraint, we obtain

$$\begin{aligned} w_{t+1} &= \beta w_t, \\ c_t &= (1 - \beta) w_t, \end{aligned}$$

implying that it is optimal to save a constant fraction of the cake and eat the remaining fraction.

Example 2: Robinson Crusoe economy with log utility

Now consider specific functional forms for the model laid out in section 1.1. Assume $u(c) = \log(c)$ and $f(k) = Ak^\alpha$, where $A > 0$ is a constant technology parameter and $\alpha \in (0, 1)$ is the cost share of capital. Also assume full depreciation, $\delta = 1$. For a given capital stock, the Bellman equation is given by

$$V(k_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta V(k_{t+1})\},$$

where the planner's choices are constrained by $c_t + k_{t+1} = Ak_t^\alpha$. After substituting for c_t using the budget constraint, the the problem reduces to

$$V(k_t) = \max_{k_{t+1}} \{\ln(Ak_t^\alpha - k_{t+1}) + \beta V(k_{t+1})\}. \tag{1.28}$$

Then if we guess that the value function has function form $V(k) = E + F \ln k$, the functional equation becomes

$$E + F \ln k_t = \max_{k_{t+1}} \{\ln(Ak_t^\alpha - k_{t+1}) + \beta(E + F \ln k_{t+1})\}. \tag{1.29}$$

The first order condition implies

$$-\frac{1}{Ak_t^\alpha - k_{t+1}} + \frac{\beta F}{k_{t+1}} = 0.$$

Thus, after simplification, we obtain

$$k_{t+1} = \frac{Ak_t^\alpha \beta F}{1 + \beta F}. \tag{1.30}$$

Plugging the result of (1.30) into the value function (1.29), we obtain

$$E + F \ln k_t = \ln\left(\frac{Ak_t^\alpha}{1 + \beta F}\right) + \beta E + \beta F \ln\left(\frac{Ak_t^\alpha \beta F}{1 + \beta F}\right).$$

Equating coefficients, we find $F = \alpha + \alpha\beta F$, which implies $F = \alpha/(1 - \alpha\beta)$. Hence

$$\begin{aligned} E &= \ln\left(\frac{A}{1 - \beta F}\right) + \beta E + \beta F \ln\left(\frac{A\beta F}{1 + \beta F}\right) \\ &= \frac{1}{1 - \beta}\left[\ln\left((1 - \alpha\beta)A\right) + \frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta A)\right]. \end{aligned}$$

Thus, plugging the value of B into (1.30) and utilizing the budget constraint, we obtain

$$\begin{aligned} k_{t+1} &= \alpha\beta Ak_t^\alpha = \alpha\beta y_t, \\ c_t &= (1 - \alpha\beta)Ak_t^\alpha = (1 - \alpha\beta)y_t. \end{aligned}$$

The results shows that the optimal policy is to save a constant fraction of output. The fact that $\alpha < 1$ implies that k_t converges as t approaches infinity for any positive initial value k_0. The stationary point is given by $k^* = (\alpha\beta A)^{1/(1-\alpha)}$.

Example 3: Robinson Crusoe economy with CRRA utility

Return to the model laid out in the previous example, but under different functional forms. Assume $u(c) = c^{1-\sigma}/(1-\sigma)$ and $f(k) = Ak$, where $\sigma = -cu''(c)/u'(c)$ is the constant of relative risk aversion. For a given capital stock, the Bellman equation is given by

$$V(k_t) = \max_{c_t, k_{t+1}} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1}) \right\},$$

where the planner's choices are constrained by $c_t + k_{t+1} = Ak_t$. After substituting for c_t using the budget constraint, the problem reduces to

$$V(w_t) = \max_{k_{t+1}} \left\{ \frac{(Ak_t - k_{t+1})^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1}) \right\}. \tag{1.31}$$

Then if we guess that the value function has function form: $V(k) = Bk^{1-\sigma}/(1-\sigma)$ the functional equation becomes

$$\frac{Bk_t^{1-\sigma}}{1-\sigma} = \max_{k_{t+1}} \left\{ \frac{(Ak_t - k_{t+1})^{1-\sigma}}{1-\sigma} + \frac{\beta B k_{t+1}^{1-\sigma}}{1-\sigma} \right\}. \tag{1.32}$$

The first order condition implies

$$(Ak_t - k_{t+1})^{-\sigma}(-1) + \beta B k_{t+1}^{-\sigma} = 0.$$

Thus, after simplification, we obtain

$$k_{t+1} = \frac{Ak_t}{1 + (\beta B)^{-1/\sigma}}. \tag{1.33}$$

Plugging (1.33) into the value function (1.32), we obtain

$$\begin{aligned}
\frac{Bk_t^{1-\sigma}}{1-\sigma} &= \frac{1}{1-\sigma} \left[\left(\frac{(\beta B)^{-1/\sigma} A k_t}{1 + (\beta B)^{-1/\sigma}} \right)^{1-\sigma} + \beta B \left(\frac{Ak_t}{1 + (\beta B)^{-1/\sigma}} \right)^{1-\sigma} \right] \\
\to \; & \left[1 + (\beta B)^{-1/\sigma}\right]^{1-\sigma} B = \left[(\beta B)^{-1/\sigma} A\right]^{1-\sigma} + \beta B A^{1-\sigma} \\
\to \; & \left[1 + (\beta B)^{-1/\sigma}\right]^{1-\sigma} = \beta A^{1-\sigma} \left[1 + (\beta B)^{-1/\sigma}\right] \\
\to \; & \left[1 + (\beta B)^{-1/\sigma}\right]^{-\sigma} = \beta A^{1-\sigma} \\
\to \; & (\beta B)^{-1/\sigma} = \beta^{-1/\sigma} A^{\frac{\sigma-1}{\sigma}} - 1 \\
\to \; & B = \frac{1}{\beta} \left[\beta^{-1/\sigma} A^{\frac{\sigma-1}{\sigma}} - 1\right]^{-\sigma}
\end{aligned}$$

Substituting the value of B into (1.33), we obtain

$$\begin{aligned}
k_{t+1} &= \frac{A}{\beta^{-1/\sigma} A^{\frac{\sigma-1}{\sigma}}} k_t \\
&= (\beta A)^{1/\sigma} k_t \\
&= \beta^{1/\sigma} A^{\frac{1-\sigma}{\sigma}} y_t,
\end{aligned}$$

and using the budget constraint

$$c_t = \left(1 - \beta^{\frac{1}{\sigma}} A^{\frac{1-\sigma}{\sigma}}\right) y_t,$$

which is the same solution given in Example 2 when $\alpha = \sigma = 1$.

Model 4: Robinson Crusoe economy with Capital Adjustment Costs

Once again, consider the Robinson Crusoe economy with log utility, full depreciation, and $f(k) = Ak^\alpha$ presented above, but with capital adjustment costs. In this case, the planner's choices are constrained by

$$c_t + i_t = Ak_t^\alpha \tag{1.34}$$

$$k_{t+1} = k_t^{1-\delta} i_t^\delta, \tag{1.35}$$

where δ measures the size of the adjustment cost. For a given capital stock, the Bellman equation is given by

$$V(k_t) = \max_{i_t, k_{t+1}} \{\ln(Ak_t^\alpha - i_t) + \beta V(k_{t+1})\}.$$

Then if we guess that the value function has function form $V(k) = E + F \ln k$ the functional equation becomes

$$\begin{aligned} V(k_t) &= \max_{i_t, k_{t+1}} \{\ln(Ak_t^\alpha - i_t) + \beta(E + F\ln(k_{t+1}))\} \\ &= \max_{i_t}\{\ln(Ak_t^\alpha - i_t) + \beta E + \beta F[(1-\delta)\ln k_t + \delta \ln(i_t)]\}. \end{aligned}$$

The first order condition implies

$$\frac{1}{Ak_t^\alpha - i_t} = \frac{\beta\delta F}{i_t} \quad \rightarrow \quad i_t = \frac{\beta\delta F A k_t^\alpha}{1 + \beta\delta F}.$$

Plugging this result back into the value function gives

$$E + F\ln k_t = \ln\left(\frac{Ak_t^\alpha}{1+\beta\delta F}\right) + \beta E + \beta F(1-\delta)\ln k_t + \beta\delta F \ln\left(\frac{\beta\delta F A k_t^\alpha}{1+\beta\delta F}\right).$$

Equate coefficients to obtain

$$F = \alpha + \beta(1-\delta)F + \alpha\beta\delta F \rightarrow F = \frac{\alpha}{1 - \beta(1-\delta) - \alpha\beta\delta}.$$

Then plug back into the first order condition to obtain

$$i_t = \frac{\alpha\beta\delta A k_t^\alpha}{1-\beta(1-\delta)} = \frac{\alpha\beta\delta}{1-\beta(1-\delta)} y_t.$$

Using the budget constraint we have

$$c_t = \left[\frac{1 - \beta(1-\delta) - \alpha\beta\delta}{1-\beta(1-\delta)}\right] y_t,$$

which is a more general solution to the basic Robinson Crusoe economy. In the case where $\delta = 1$, $k_{t+1} = i_t$ and the solution is identical one derived in Example 2.

1.4.2 Value Function Iteration

When the intertemporal problem in discrete time has a time-separable objective function that can be represented recursively, it can be solved using the "principle of optimality" due to Bellman (1957). This method is known as *dynamic programming*. The basic idea of the principle of optimality is to solve the optimization problem period-by-period—starting with the final period, taking the previous periods' solutions as given, and then working back sequentially to find the first period.

Once again, consider the Robinson Crusoe economy with log utility, full depreciation, and $f(k) = Ak^\alpha$ presented earlier. Start with an initial guess for the value function at time 0: $V_0(k) = 0$. This guess acts as a starting point, which is the same role a state variable plays. Using the budget constraint and this guess, we have

$$\begin{aligned} V_1(k) &= \max_{k'}\{\ln(Ak^\alpha - k') + \beta V_0(k')\} \\ &= \max_{k'}\{\ln(Ak^\alpha - k')\}. \end{aligned}$$

Since the log function is increasing, the maximum occurs at $k' = 0$. Thus, we have

$$\begin{aligned} V_1(k) &= \ln(Ak^\alpha) = \ln A + \alpha \ln k \\ c &= Ak^\alpha \end{aligned}$$

Continuing to iterate, we obtain

$$\begin{aligned} V_2(k) &= \max_{k'}\{\ln(Ak^\alpha - k') + \beta V_1(k')\} \\ &= \max_{k'}\{\ln(Ak^\alpha - k') + \beta(\ln A + \alpha \ln k')\}. \end{aligned} \tag{1.36}$$

The first order condition implies

$$\frac{1}{Ak^\alpha - k'}(-1) + \frac{\alpha\beta}{k'} = 0.$$

After simplification, we obtain

$$k' = \frac{\alpha\beta Ak^\alpha}{1+\alpha\beta} \qquad \text{and} \qquad c = \frac{Ak^\alpha}{1+\alpha\beta}$$

Thus, plugging the value of k' into (1.36), we obtain

$$\begin{aligned} V_2(k) &= \ln\left(\frac{Ak^\alpha}{1+\alpha\beta}\right) + \beta\ln A + \alpha\beta\ln\left(\frac{\alpha\beta Ak^\alpha}{1+\alpha\beta}\right) \\ &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta\ln A + \alpha\beta\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln k \end{aligned}$$

Continuing to iterate, we obtain

$$\begin{aligned} V_3(k) &= \max_{k'} \ \left\{\ln(Ak^\alpha - k') + \beta V_2(k')\right\} \\ &= \max_{k'} \ \left\{\ln(Ak^\alpha - k') + \beta\left[\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta\ln A \right.\right. \\ &\qquad \left.\left. + \ \alpha\beta\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln k'\right]\right\} \end{aligned} \tag{1.37}$$

The first order condition implies

$$\frac{1}{Ak^{\alpha}-k'}(-1)+\frac{\alpha\beta(1+\alpha\beta)}{k'}=0.$$

Thus, after simplification, we obtain

$$k'=\frac{\alpha\beta(1+\alpha\beta)Ak^{\alpha}}{1+\alpha\beta(1+\alpha\beta)} \qquad \text{and} \qquad c=\frac{Ak^{\alpha}}{1+\alpha\beta(1+\alpha\beta)}$$

Thus, plugging the value of k' into (1.37), we obtain

$$\begin{aligned}V_3(k)&=\ln\left(\frac{Ak^{\alpha}}{1+\alpha\beta(1+\alpha\beta)}\right)+\beta\left[\beta\ln A+\ln\left(\frac{A}{1+\alpha\beta}\right)+\alpha\beta\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right)\right]+\\&\quad\alpha\beta(1+\alpha\beta)\ln\left(\frac{\alpha\beta(1+\alpha\beta)Ak^{\alpha}}{1+\alpha\beta(1+\alpha\beta)}\right)\\&=\text{constant}+\alpha\left(1+\alpha\beta+(\alpha\beta)^2\right)\ln k\end{aligned}$$

Since $\alpha\beta<1$, continuing to iterate, we eventually obtain

$$\begin{aligned}V(k)&=\max_{k'}\ \left\{\ln(Ak^{\alpha}-k')+\beta V(k')\right\}\\&=\max_{k'}\ \left\{\ln(Ak^{\alpha}-k')+\beta\left[\text{constant}+\frac{\alpha}{1-\alpha\beta}\ln k'\right]\right\}\end{aligned}$$

The first order condition implies

$$\frac{1}{Ak^{\alpha}-k'}(-1)+\frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k'}=0.$$

Thus, after simplification, we obtain

$$\begin{aligned}k'&=\alpha\beta Ak^{\alpha}=\alpha\beta y\\c&=Ak^{\alpha}(1-\alpha\beta)=(1-\alpha\beta)y,\end{aligned}$$

which is the same solution that we arrived at in Example 2 in section 1.4.1.

1.4.3 Euler Equation Iteration

Example 1: Robinson Crusoe economy with log utility

Once again, consider the Robinson Crusoe economy with log utility, full depreciation, and $y_t = f(k)=Ak^{\alpha}$ presented above. From equation (1.6), we obtain the following Euler equation

$$\frac{1}{c_t}=\frac{1}{c_{t+1}}A\alpha\beta k_{t+1}^{\alpha-1}=\alpha\beta\frac{1}{c_{t+1}}\frac{y_{t+1}}{k_{t+1}}.$$

Using the budget constraint and adding 1 to both sides, it follows that

$$\frac{c_t+k_{t+1}}{c_t}=1+\alpha\beta\frac{c_{t+1}+k_{t+2}}{c_{t+1}} \tag{1.38}$$

Defining $z_t = \frac{c_t+k_{t+1}}{c_t} = \frac{y_t}{c_t}$ and iterating, we get

$$\begin{aligned} z_t &= 1 + \alpha\beta z_{t+1} \\ &= 1 + \alpha\beta + (\alpha\beta)^2 z_{t+2} \\ &\vdots \\ &= \sum_{t=0}^{\infty} (\alpha\beta)^t + \lim_{T\to\infty} (\alpha\beta)^T z_{t+T} \\ &= \frac{1}{1-\alpha\beta}, \end{aligned}$$

provided that $\alpha\beta < 1$ and the transversality condition holds. Using the definition of z_t, it is easy to see that

$$\begin{aligned} c_t &= (1-\alpha\beta) y_t \\ k_{t+1} &= \alpha\beta y_t, \end{aligned}$$

which is the same solution that we arrived at in Example 2 in section 1.4.1.

Example 2: Robinson Crusoe economy with Capital Adjustment Costs

Now consider the Robinson Crusoe economy with capital adjustment costs studied in Example 4 of section 1.4.1. The Lagrangian is given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{\ln c_t + \lambda_t (A k_t^\alpha - c_t - i_t) + \mu_t (k_t^{1-\delta} i_t^\delta - k_{t+1})\}$$

where λ_t is the Lagrange multiplier on (1.34) and μ_t is the Lagrange multiplier on (1.35). The first order conditions imply

$$\begin{aligned} c_t : &\quad \frac{1}{c_t} = \lambda_t \\ i_t : &\quad \lambda_t = \mu_t k_t^{1-\delta} \delta i_t^{\delta-1} = \frac{\delta \mu_t k_{t+1}}{i_t} \quad \rightarrow \quad \mu_t = \frac{i_t}{\delta c_t k_{t+1}} \\ k_{t+1} : &\quad \mu_t = \beta \lambda_{t+1} \alpha A k_{t+1}^{\alpha-1} + \beta \mu_{t+1} (1-\delta) k_{t+1}^{-\delta} i_{t+1}^{\delta} = \frac{\alpha\beta\lambda_{t+1} y_{t+1}}{k_{t+1}} + \frac{\beta(1-\delta)\mu_{t+1} k_{t+2}}{k_{t+1}}. \end{aligned}$$

Combine the first order conditions to obtain

$$\frac{i_t}{c_t} = \alpha\beta\delta \frac{y_{t+1}}{c_{t+1}} + \beta(1-\delta)\frac{i_{t+1}}{c_{t+1}}.$$

Using the budget constraint and a bit of algebra gives

$$\frac{c_t + i_t}{c_t} = 1 - \beta(1-\delta) + [\alpha\beta\delta + \beta(1-\delta)] \left(\frac{c_{t+1} + i_{t+1}}{c_{t+1}} \right)$$

Iterating and applying the transversality condition yields

$$\frac{y_t}{c_t} = [1-\beta(1-\delta)] \sum_{i=0}^{\infty} [\alpha\beta\delta + \beta(1-\delta)]^i = \frac{1-\beta(1-\delta)}{1-\alpha\beta\delta-\beta(1-\delta)} \equiv \eta$$

Thus, using the budget constraint, we obtain

$$c_t = y_t/\eta,$$
$$i_t = (1 - 1/\eta)y_t$$

which is the same solution we derived in in Example 4 in section 1.4.1.

Example 3: Robinson Crusoe economy with Capital Adjustment Costs and Elastic Labor

Now consider a similar example, but with an elastic labor supply. In this case, the planner chooses sequences $\{c_t, n_t, k_t, i_t\}$ to maximize lifetime utility, given by,

$$\sum_{t=0}^{\infty} \beta^t \{\ln c_t + \gamma(1 - n_t)\},$$

where n_t is hours worked. These choices are constrained by

$$c_t + i_t = y_t \equiv Ak_{t-1}^{\alpha} n_t^{1-\alpha} \tag{1.39}$$
$$k_t = h(i_t, k_{t-1}) \equiv i_t^{\delta} k_{t-1}^{1-\delta}, \tag{1.40}$$

where $h(\cdot)$ is the law of motion for the capital stock. The Lagrangian is given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{\ln c_t + \gamma(1 - n_t) + \lambda_t (Ak_{t-1}^{\alpha} n_t^{1-\alpha} - c_t - i_t) + \mu_t (i_t^{\delta} k_{t-1}^{1-\delta} - k_t)\},$$

where λ_t is the Lagrange multiplier on (1.39) and μ_t is the Lagrange multiplier on (1.40). The first order conditions imply

$$\begin{aligned}
c_t &: \quad \frac{1}{c_t} = \lambda_t \\
n_t &: \quad \gamma = \lambda_t A k_{t-1}^{\alpha}(1-\alpha) n_t^{-\alpha} \\
i_t &: \quad \lambda_t = \mu_t \delta i_t^{\delta-1} k_{t-1}^{1-\delta} \\
k_t &: \quad \mu_t = \beta \lambda_{t+1} A \alpha k_t^{\alpha-1} n_{t+1}^{1-\alpha} + \beta \mu_{t+1} i_{t+1}^{\delta} (1-\delta) k_t^{-\delta}.
\end{aligned}$$

Eliminating λ_t and simplifying yields

$$\begin{aligned}
\gamma &= \frac{(1-\alpha)y_t}{c_t n_t} \\
\mu_t &= \frac{i_t}{\delta c_t k_t} \\
\mu_t &= \frac{\alpha\beta y_{t+1}}{c_{t+1} k_t} + \frac{\beta(1-\delta)\mu_{t+1} k_{t+1}}{k_t}.
\end{aligned}$$

Combining the two equations for μ_t implies

$$\frac{i_t}{\delta c_t} = \frac{\alpha\beta y_{t+1}}{c_{t+1}} + \frac{\beta(1-\delta) i_{t+1}}{\delta c_{t+1}}.$$

Manipulating this equation into a difference equation then yields

$$\frac{i_t + c_t}{c_t} = 1 - \beta(1-\delta) + \beta\delta \left[\left(\alpha + \frac{1-\delta}{\delta} \right) \left(\frac{c_{t+1} + i_{t+1}}{c_{t+1}} \right) \right].$$

Iterating and applying the transversality condition yields

$$\begin{aligned}\frac{y_t}{c_t} = \frac{i_t + c_t}{c_t} &= \sum_{j=0}^{\infty}\left[\beta\delta\left(\alpha + \frac{1-\delta}{\delta}\right)\right]^j (1-\beta(1-\delta)) \\ &= \sum_{j=0}^{\infty}[\beta(\alpha\delta + 1 - \delta)]^j(1-\beta(1-\delta)) \\ &= \frac{1-\beta(1-\delta)}{1-\alpha\beta\delta-\beta(1-\delta)} \equiv \eta.\end{aligned}$$

Thus, we have the following solution paths

$$\begin{aligned}c_t &= y_t/\eta \\ i_t &= (1-1/\eta)y_t \\ n_t &= (1-\alpha)\eta/\gamma,\end{aligned}$$

which is the same solution we arrived at in the previous example, except that we now have an optimal labor choice.

Example 4: Robinson Crusoe economy with Habit Persistence

Now return to the basic Robinson Crusoe economy, but add habit persistence. In this case, the planner chooses sequences $\{c_t, k_t, i_t\}$ to maximize lifetime utility, given by,

$$\sum_{t=0}^{\infty}\beta^t\{\ln c_t + \gamma \ln c_{t-1}\},$$

where $\gamma > 0$ measures the degree of habit persistence. This utility function is referred to as habit persistence because last period's consumption enters this period's utility function discounted (if $\gamma \in (0,1)$) or at a premium (if $\gamma > 1$). For this reason, notice that time separable utility does not hold here. However, we could defined a change of variable $x_t = c_t c_{t-1}^{\gamma}$ that leads to the more common $u(x_t) = \ln(x_t)$.

The planner's choices are constrained by

$$c_t + k_{t+1} = Ak_t^{\alpha}.$$

For a given reference level of consumption and capital stock, the Bellman equation is given by

$$\begin{aligned}V(c_{t-1}, k_t) &= \max_{c_t, k_{t+1}}\{\ln c_t + \gamma \ln c_{t-1} + \beta V(c_t, k_{t+1})\} \\ &= \max_{k_{t+1}}\{\ln(Ak_t^{\alpha} - k_{t+1}) + \gamma \ln c_{t-1} + \beta V(Ak_t^{\alpha} - k_{t+1}, k_{t+1})\}.\end{aligned}$$

The first order condition implies

$$\frac{1}{Ak_t^{\alpha} - k_{t+1}} + \beta V_1(c_t, k_{t+1}) = \beta V_2(c_t, k_{t+1}).$$

and the envelope conditions yield

$$\begin{aligned}V_2(c_{t-1}, k_t) &= \frac{\alpha A k_t^{\alpha-1}}{Ak_t^{\alpha} - k_{t+1}} + \alpha\beta V_1(c_t, k_{t+1})Ak_t^{\alpha-1} \\ V_1(c_{t-1}, k_t) &= \frac{\gamma}{c_{t-1}}.\end{aligned}$$

Combining the two envelope conditions and updating yields

$$V_2(c_t, k_{t+1}) = \frac{\alpha(1+\beta\gamma)y_{t+1}}{c_{t+1}k_{t+1}}.$$

Plug this result back into the first order condition to obtain

$$\frac{1}{c_t} = \frac{\alpha\beta y_{t+1}}{c_{t+1}k_{t+1}},$$

which implies that

$$\frac{c_t + k_{t+1}}{c_t} = 1 + \alpha\beta\left(\frac{c_{t+1}+k_{t+2}}{c_{t+1}}\right).$$

Thus, after iterating and applying the transversality condition, we have the following solutions

$$\begin{aligned} c_t &= (1-\alpha\beta)y_t \\ k_{t+1} &= \alpha\beta y_t \end{aligned}$$

Another way to see that this Euler equation will be equivalent to the one without habit persistence is to note that

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t\{\ln c_t + \gamma\ln c_{t-1}\} &= \sum_{t=0}^{\infty}\beta^t\ln c_t + \gamma\sum_{t=0}^{\infty}\beta^t\ln c_{t-1} \\ &= \sum_{t=0}^{\infty}\beta^t\ln c_t + \gamma\sum_{t=0}^{\infty}\beta^{t+1}\ln c_t + \gamma\ln c_{-1} \\ &= (1+\beta\gamma)\sum_{t=0}^{\infty}\beta^t\ln c_t + \gamma\ln c_{-1}. \end{aligned}$$

Since c_{-1} is given, the problem is identical to the Robinson Crusoe economy without habit persistence. The coefficient $1+\beta\gamma$ just scales the utility function.

1.4.4 Howard's Improvement Algorithm

Example 1: Robinson Crusoe economy with log utility

Once again, consider the Robinson Crusoe economy with log utility, full depreciation, and $y_t = f(k) = Ak^\alpha$ presented above. We have already seen that the Euler equation is given by

$$\frac{1}{c_t} = \frac{1}{c_{t+1}}A\alpha\beta k_{t+1}^{\alpha-1} = \alpha\beta\frac{1}{c_{t+1}}\frac{y_{t+1}}{k_{t+1}},$$

which implies

$$c_t = \frac{1}{\alpha\beta}\frac{c_{t+1}k_{t+1}}{y_{t+1}}$$

Next, guess that the policy function is given by $c_t = \theta y_t$, where θ is some unknown constant. Then the Euler equation becomes

$$\theta y_t = \frac{1}{\alpha\beta}\frac{\theta y_{t+1}k_{t+1}}{y_{t+1}}.$$

After simplifying, it is easy to see that

$$\begin{aligned} k_{t+1} &= \alpha\beta y_t \\ c_t &= (1-\alpha\beta)y_t, \end{aligned}$$

which is the same solution that we arrived at in Example 2 in section 1.4.1..

Example 2: Robinson Crusoe economy with CRRA utility

Return to the model laid out in the previous example, but under different functional forms. Assume $u(c) = c^{1-\sigma}/(1-\sigma)$ and $f(k) = Ak$. Given the general form of the Euler equation in (1.6), the Euler equation under CRRA utility is given by

$$c_t^{-\sigma} = \beta A c_{t+1}^{-\sigma},$$

which implies

$$c_t = (\beta A)^{-1/\sigma} c_{t+1}.$$

Once again, guess that $c_t = \theta y_t$. Then the Euler equation becomes

$$\theta y_t = (\beta A)^{-1/\sigma} \theta y_{t+1},$$

which, after substituting in the production function, becomes

$$\theta A k_t = (\beta A)^{-1/\sigma} \theta A k_{t+1}.$$

After simplifying, the period-to-period capital ratio is given by

$$\frac{k_t}{k_{t+1}} = (\beta A)^{-1/\sigma}. \tag{1.41}$$

Also, substituting the policy function guess, $c_t = \theta y_t$, into the budget constraint, we obtain

$$\theta y_t + k_{t+1} = A k_t \qquad \to \qquad \theta A k_t + k_{t+1} = A k_t,$$

which implies that the period-to-period capital ratio is given by

$$\frac{k_{t+1}}{k_t} = (A - \theta A). \tag{1.42}$$

Combining (1.41) and (1.42), we obtain the unknown policy function coefficient, given by,

$$\begin{aligned} & (\beta A)^{1/\sigma} = A - \theta A \\ \to \quad & \theta = \frac{A - (\beta A)^{1/\sigma}}{A} \\ \to \quad & \theta = 1 - \beta^{1/\sigma} A^{(1-\sigma)/\sigma}. \end{aligned}$$

Substituting for θ in our policy guess, we obtain

$$\begin{aligned} c_t &= (1 - \beta^{1/\sigma} A^{(1-\sigma)/\sigma}) y_t, \\ k_{t+1} &= \beta^{1/\sigma} A^{(1-\sigma)/\sigma} y_t, \end{aligned}$$

which is the same solution that we arrived at in Example 3 in section 1.4.1. However, it is clear that this method is algebraically less tedious.

1.5 Stochastic Economy

Up to this point, out models have been deterministic. The values of all parameters of the model and the form of the function are known with certainty. Given some initial condition, these economies follow a prescribed path.

Consider a version of the infinite horizon, Robinson Crusoe economy laid out in section 1.1 where technology is stochastic. The social planner chooses sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize *expected* lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t, \tag{1.43}$$

where z_t is a stochastic technology factor and E_t is the expectation operator conditional on information at time t. The stochastic process for technology is given by

$$z_t = \bar{z}(z_{t-1}/\bar{z})^{\rho_z} \exp(\varepsilon_{z,t}),$$

where $\bar{z} > 0$ is steady state productivity, $0 \leq \rho_z < 1$, and $\varepsilon_{z,t} \sim \mathbb{N}(0, \sigma_z^2)$. Note that c_t is known in period t, but c_{t+i}, for $i = 1, 2, \ldots$ is unknown. That is, expected utility at the beginning of period $t+1$ is uncertain as of the beginning of period t.

Mapping this problem into a dynamic programming problem is a fairly straightforward generalization of the case under certainty. The relevant state variables are k_t and z_t. The Bellman equation is given by

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta E_t V(k_{t+1}, z_{t+1})\}$$

subject to (1.43).

1.5.1 Example: Stochastic Technology

Our goal is to determine the value function $V(\cdot,\cdot)$ and the optimal decision rules for the choice variables (i.e. $k_{t+1} = g(k_t, z_t)$ and $c_t = z_t f(k_t) + (1-\delta)k_t - g(k_t, z_t)$). As an example, let $f(k_t) = k_t^{\alpha}$, with $0 < \alpha < 1$, $u(c_t) = \ln c_t$, $\delta = 1$, and $\rho_z = 0$. Guess that the value function takes the form $V(k_t, z_t) = A + B \ln k_t + D \ln z_t$. Then the Bellman equation can be written

$$\begin{aligned} A + B\ln k_t + D\ln z_t &= \max_{k_{t+1}} \{\ln(z_t k_t^{\alpha} - k_{t+1}) + \beta E_t[A + B\ln k_{t+1} + D\ln z_{t+1}]\} \\ &= \max_{k_{t+1}} \{\ln(z_t k_t^{\alpha} - k_{t+1}) + \beta A + \beta B E_t[\ln k_{t+1}] + \beta D\tilde{z}\}, \end{aligned}$$

where $\tilde{z} = E_t[\ln z_{t+1}]$. Solving the optimization problem on the right-hand side of the above equation gives

$$k_{t+1} = \frac{\beta B}{1+\beta B} z_t k_t^{\alpha}. \tag{1.44}$$

Substituting into the Bellman equation yields

$$A + B\ln k_t + D\ln z_t = \ln\left(\frac{z_t k_t^{\alpha}}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B z_t k_t^{\alpha}}{1+\beta B}\right) + \beta D\tilde{z}$$

Our guess is verified if there exists a solution for A, B, and D. Equating coefficients on either side of the above equation gives

$$\begin{aligned} A &= \ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right) + \beta D \tilde{z}, \\ B &= \alpha + \alpha\beta B, \\ D &= 1 + \beta B. \end{aligned}$$

Solving this system of equations implies

$$\begin{aligned} B &= \frac{\alpha}{1-\alpha\beta}, \\ D &= \frac{1}{1-\alpha\beta}, \\ A &= \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \frac{\beta\mu}{1-\alpha\beta}\right]. \end{aligned}$$

Substituting into (1.44) gives the policy functions

$$k_{t+1} = \alpha\beta z_t k_t^{\alpha}, \tag{1.45}$$

$$c_t = (1-\alpha\beta) z_t k_t^{\alpha}, \tag{1.46}$$

which is the same solution that we arrived at in Example 2 in section 1.4.1 except that z_t is stochastic. Thus, this economy will *not* converge to a steady state, since technology shocks (z_t) will cause persistent fluctuations in output, consumption, and investment.

Chapter 2

Linear Discrete Time Models

2.1 Analytical Solution Methods

2.1.1 Model Setup

Consider a standard decentralized economy where the consumer chooses sequences $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \log(C_t)$$

subject to

$$C_t + K_t = (1 - \tau_t) R_t K_{t-1} + T_t + \Pi_t,$$

where τ is a proportional tax levied on income, R is the gross rental price of capital, T is a lump sum transfer from the government, and Π_t is the representative firm's profits that are rebated back to the consumer. The first order conditions are given by

$$\frac{1}{C_t} = \lambda_t$$
$$\lambda_t = \beta E_t\{\lambda_{t+1}(1 - \tau_{t+1}) R_{t+1}\},$$

where λ is the Lagrange multiplier on the consumer's budget constraint. Combining these results yields the following Euler equation

$$\frac{1}{C_t} = \beta E_t \left\{ \frac{1}{C_{t+1}} (1 - \tau_{t+1}) R_{t+1} \right\}. \tag{2.1}$$

Each period the competitive firm chooses K_t to maximize profits, given by,

$$\Pi_t = Y_t - R_t K_{t-1}$$

subject to the output constraint, $Y_t = A_t K_{t-1}^{\alpha}$, where Y is output and A is an exogenous i.i.d mean zero technology shock. The firm's first order condition then implies

$$R_t = \alpha A_t K_{t-1}^{\alpha - 1}. \tag{2.2}$$

Combining (2.1) and (2.2) yields

$$\begin{aligned}\frac{1}{C_t} &= \beta E_t\left\{(1-\tau_{t+1})\frac{1}{C_{t+1}}\alpha A_{t+1}K_t^{\alpha-1}\right\}\\ &= \alpha\beta E_t\left\{(1-\tau_{t+1})\frac{1}{C_{t+1}}\frac{Y_{t+1}}{K_t}\right\}.\end{aligned} \tag{2.3}$$

The aggregate resource constraint is given by

$$C_t + K_t = Y_t = A_t K_{t-1}^{\alpha}, \tag{2.4}$$

which is verified by combining the consumer's budget constraint and the government's budget constraint, given by,

$$T_t = \tau_t R_t K_{t-1}$$

and noting that $\Pi_t = (1-\alpha)Y_t$.

2.1.2 Log-Linear System

Log-linearizing (2.3) yields

$$\begin{aligned}-\frac{1}{C^2}Cc_t &= \alpha\beta(1-\tau)\frac{A}{C}(\alpha-1)K^{\alpha-2}Kk_t + \alpha\beta(1-\tau)\frac{K^{\alpha-1}}{C}AE_t a_{t+1}\\ &\quad - \alpha\beta(1-\tau)AK^{\alpha-1}\frac{1}{C^2}CE_t c_{t+1} - \alpha\beta\frac{AK^{\alpha-1}}{C}\tau E_t\hat{\tau}_{t+1}\\ &= \alpha\beta(1-\tau)\frac{AK^{\alpha-1}}{C}\left[(\alpha-1)k_t + E_t a_{t+1} - E_t c_{t+1} - \frac{\tau}{1-\tau}E_t\hat{\tau}_{t+1}\right],\end{aligned}$$

which after making use of its steady state condition reduces to

$$-c_t = (\alpha-1)k_t + E_t a_{t+1} - E_t c_{t+1} - \frac{\tau}{1-\tau}E_t\hat{\tau}_{t+1}. \tag{2.5}$$

Linearizing (2.4) gives

$$\begin{aligned}Cc_t + Kk_t &= A\alpha K^{\alpha-1}Kk_{t-1} + K^{\alpha}Aa_t\\ &= AK^{\alpha-1}K(\alpha k_{t-1} + a_t)\\ &= \frac{K}{\alpha\beta(1-\tau)}(\alpha k_{t-1} + a_t),\end{aligned}$$

where the last line follows from imposing steady state on (2.3). Using the fact that

$$\frac{C}{K} = AK^{\alpha-1} - 1 = \frac{1}{\alpha\beta(1-\tau)} - 1 = \frac{1-\alpha\beta(1-\tau)}{\alpha\beta(1-\tau)},$$

the above result can be rewritten as

$$\begin{aligned}&\frac{C}{K}c_t = \frac{1}{\alpha\beta(1-\tau)}(\alpha k_{t-1} + a_t) - k_t\\ \rightarrow\ &\left[\frac{1-\alpha\beta(1-\tau)}{\alpha\beta(1-\tau)}\right]c_t = \frac{1}{\alpha\beta(1-\tau)}(\alpha k_{t-1} + a_t) - k_t\\ \rightarrow\ &c_t = \frac{1}{1-\alpha\beta(1-\tau)}a_t + \frac{\alpha}{1-\alpha\beta(1-\tau)}k_{t-1} - \frac{\alpha\beta(1-\tau)}{1-\alpha\beta(1-\tau)}k_t\end{aligned} \tag{2.6}$$

Thus, the forecast error in consumption can be written

$$\begin{aligned} c_t - E_t c_{t+1} &= \frac{1}{1-\alpha\beta(1-\tau)}(a_t - E_t a_{t+1}) + \frac{\alpha}{1-\alpha\beta(1-\tau)}(k_{t-1} - k_t) - \frac{\alpha\beta(1-\tau)}{1-\alpha\beta(1-\tau)}(k_t - E_t k_{t+1}) \\ &= \frac{1}{1-\alpha\beta(1-\tau)}(a_t - E_t a_{t+1}) + \frac{\alpha}{1-\alpha\beta(1-\tau)}k_{t-1} - \left[\frac{\alpha+\alpha\beta(1-\tau)}{1-\alpha\beta(1-\tau)}\right]k_t + \frac{\alpha\beta(1-\tau)}{1-\alpha\beta(1-\tau)}E_t k_{t+1}. \end{aligned}$$

Rearranging (2.5), the forecast error in consumption can be equivalently written as

$$c_t - E_t c_{t+1} = \frac{\tau}{1-\tau}E_t\hat{\tau}_{t+1} + (1-\alpha)k_t - E_t a_{t+1}.$$

Combining the previous two results and multiplying by the coefficient on $E_t k_{t+1}$ yields

$$\begin{aligned} &\frac{1-\alpha\beta(1-\tau)}{\alpha\beta}\frac{\tau}{1-\tau}E_t\hat{\tau}_{t+1} + \frac{[1-\alpha\beta(1-\tau)](1-\alpha)}{\alpha\beta}k_t - \frac{1-\alpha\beta(1-\tau)}{\alpha\beta}E_t a_{t+1} \\ &= \frac{1}{\alpha\beta}(a_t - E_t a_{t+1}) + \frac{1}{\beta}k_{t-1} - \frac{\alpha+\alpha\beta(1-\tau)}{\alpha\beta}k_t + (1-\tau)E_t k_{t+1} \end{aligned}$$

which, after simplifying, implies

$$(1-\tau)E_t k_{t+1} - \frac{1+\alpha^2\beta(1-\tau)}{\alpha\beta}k_t + \frac{1}{\beta}k_{t-1} = \frac{1-\alpha\beta(1-\tau)}{\alpha\beta}\frac{\tau}{1-\tau}E_t\hat{\tau}_{t+1} + (1-\tau)E_t a_{t+1} - \frac{1}{\alpha\beta}a_t.$$

After dividing by $1-\tau$ yields

$$E_t k_{t+1} - (\theta^{-1}+\alpha)k_t + \alpha\theta^{-1}k_{t-1} = \theta^{-1}(1-\theta)\left(\frac{\tau}{1-\tau}\right)E_t\hat{\tau}_{t+1} + E_t[a_{t+1} - \theta^{-1}a_t],$$

where $\theta \equiv \alpha\beta(1-\tau)$. Written more compactly,we obtain

$$E_t k_{t+1} - \gamma_0 k_t + \gamma_1 k_{t-1} = \nu_2 a_t + \nu_1 E_t\hat{\tau}_{t+1} \equiv E_t x_t, \tag{2.7}$$

where

$$\begin{aligned} \gamma_0 &= \theta^{-1} + \alpha \qquad \nu_1 = \theta^{-1}(1-\theta)\left(\frac{\tau}{1-\tau}\right) \\ \gamma_1 &= \alpha\theta^{-1} \qquad \nu_2 = -\theta^{-1}. \end{aligned}$$

Thus, the equilibrium is characterized by a second order difference equation in capital.

2.1.3 Solution Method I: Direct Approach

Using lag operators, (2.7) can be written as

$$\begin{aligned} E_t x_t &= E_t(L^{-2} - \gamma_0 L^{-1} + \gamma_1)k_{t-1} \\ &= E_t(\lambda_1 - L^{-1})(\lambda_2 - L^{-1})k_{t-1}, \end{aligned}$$

where

$$\gamma_1 = \lambda_1\lambda_2 = \alpha\theta^{-1} \qquad \gamma_0 = \lambda_1 + \lambda_2 = \theta^{-1} + \alpha.$$

Thus, it is easy to see that $\lambda_1 = \alpha < 1$ and $\lambda_2 = \theta^{-1} = [\alpha\beta(1-\tau)]^{-1} > 1$. Inverting the unstable factor, $\lambda_2 - L^{-1}$, in the above equation then yields

$$\begin{aligned}
k_t &= \lambda_1 k_{t-1} - (\lambda_2 - L^{-1})^{-1} E_t x_t \\
&= \lambda_1 k_{t-1} - \frac{\lambda_2^{-1}}{1-(\lambda_2 L)^{-1}} E_t x_t \\
&= \lambda_1 k_{t-1} - \lambda_2^{-1} \sum_{j=0}^{\infty} (\lambda_2 L)^{-j} E_t x_t \\
&= \alpha k_{t-1} - \alpha\beta(1-\tau) \sum_{j=0}^{\infty} [\alpha\beta(1-\tau)]^j E_t x_{t+j} \\
&= \alpha k_{t-1} - \sum_{j=0}^{\infty} [\alpha\beta(1-\tau)]^{j+1} E_t \left[\theta^{-1}(1-\theta)\left(\frac{\tau}{1-\tau}\right)\hat{\tau}_{t+1+j} - \theta^{-1} a_{t+j}\right] \\
&= \alpha k_{t-1} + a_t - \nu_1 \sum_{j=1}^{\infty} \theta^j E_t \hat{\tau}_{t+j}. \qquad (2.8)
\end{aligned}$$

Note that if $\tau_t = 0$ for all t, the solution, in levels, is identical to the solution given in (1.45). In logs, (2.1) in steady state implies $\ln(\alpha\beta) = (1-\alpha)\ln K - \ln A$. Hence, (2.8) in levels implies

$$\ln K_t = \ln(\alpha\beta) + \alpha \ln K_{t-1} + \ln A_t.$$

Applying the exponential functions, we obtain

$$K_t = \alpha\beta K_{t-1}^{\alpha} A_t,$$

which is the same solution given in (1.45).

Extension Now assume tax policy evolves according to a first-order two-state Markov chain with transition matrix

$$P = \begin{bmatrix} \Pr[s_t = 1|s_{t-1} = 1] & \Pr[s_t = 2|s_{t-1} = 1] \\ \Pr[s_t = 1|s_{t-1} = 2] & \Pr[s_t = 2|s_{t-1} = 2] \end{bmatrix} = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix},$$

where $0 \leq p_{ii} \leq 1$ for all $i \in \{1, 2\}$. The tax rate is given by

$$\hat{\tau}_t = \begin{cases} \hat{\tau}_{1t}, & \text{if } s_t = 1; \\ \hat{\tau}_{2t}, & \text{if } s_t = 2, \end{cases}$$

where $\hat{\tau}_{1t}$ and $\hat{\tau}_{2t}$ are low and high tax policies, whose realizations are contingent upon the state of the economy at time t, s_t.

Suppose that $p_{11} = 1$, so that the transition matrix is lower triangular. Then, once the process enters state 1, there is no possibility of ever returning to state 2. In such a case, we would say that state 1 is an *absorbing state* and that the Markov chain is *reducible*. A Markov chain that is not reducible is said to be *irreducible*. Thus, this two-state Markov chain is irreducible if $p_{ii} < 1$.

For any N-state Markov chain every row of the transition matrix P must sum to unity. Hence $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes an $N \times 1$ vector of ones. This implies that unity is an eigenvalue of the matrix P and that $\mathbf{1}$ is an associated eigenvector. Consider an N-state irreducible Markov chain with transition matrix P. Suppose that one of the eigenvalues of P is unity and that all other eigenvalues of P are inside the unit circle. Then the Markov chain is said the be *ergodic*. The $N \times 1$ vector of ergodic probabilities is denoted by π. This vector π is defined as the eigenvector of P' associated with the unit eigenvector; that is, the vector of ergodic probabilities satifies $P'\pi = \pi$ (or $(P' - I)\pi = \mathbf{0}$). The eigenvector π is normalized so that its elements sum to unity ($\mathbf{1}'\pi = 1$). It can be shown that if P is the transition matrix for an ergodic Markov chain, then $\lim_{m\to\infty} P^m = \pi\mathbf{1}'$. See Hamilton (1994) for details.

The eigenvalues of the transition matrix P for any N-state Markov chain are found from the solutions to $|P - \lambda I_N| = 0$. For the 2-state Markov chain, given above, the eigenvalues satisfy

$$\begin{aligned}
\det(P - \lambda I) &= \det\left(\begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{12} & p_{12} \end{bmatrix}\right) \\
&= (p_{11} - \lambda)(p_{22} - \lambda) - (1 - p_{11})(1 - p_{22}) \\
&= \lambda^2 - (p_{11} + p_{22})\lambda - 1 + p_{22} + p_{11} \\
&= (\lambda - 1)(\lambda + 1 - p_{11} + p_{22}) = 0.
\end{aligned}$$

Thus, the eigenvalues for a two-state chain are given by $\lambda_1 = 1$ and $\lambda_2 = -1 + p_{11} + p_{22}$. The second eigenvalue will be inside the unit circle as long as $0 < p_{11} + p_{22} < 2$. We saw earlier that this chain is irreducible as long as $p_{11} < 1$ and $p_{22} < 1$. Thus, a two-state Markov chain is ergodic provided that $p_{11} < 1$, $p_{22} < 1$, and $p_{11} + p_{22} > 0$. To solve for the eigenvector associated with $\lambda_1 = 1$ (i.e. the ergodic probabilities), note that

$$\begin{aligned}
(P' - I)\pi &= \begin{bmatrix} p_{11} - 1 & 1 - p_{22} \\ 1 - p_{11} & p_{22} - 1 \end{bmatrix} \pi = 0 \\
\rightarrow \ & \begin{bmatrix} 1 & \frac{1-p_{22}}{p_{11}-1} \\ 0 & 0 \end{bmatrix} \pi = 0 \\
\rightarrow \ & \pi_1 + \frac{1 - p_{22}}{p_{11} - 1}\pi_2 = 1 - \pi_2 + \frac{1 - p_{22}}{p_{11} - 1}\pi_2 = 0 \\
\rightarrow \ & p_{11} - 1 - (p_{11} - 1)\pi_2 + (1 - p_{22})\pi_2 = 0.
\end{aligned}$$

Hence, the ergodic probabilities are given by,

$$\pi = \begin{bmatrix} \Pr[s_t = 1] \\ \Pr[s_t = 2] \end{bmatrix} = \begin{bmatrix} \frac{1-p_{22}}{2-p_{11}-p_{22}} \\ \frac{1-p_{11}}{2-p_{11}-p_{22}} \end{bmatrix}.$$

Now return to our economic model. Given the Markov process P, the conditional expectations are given by

$$\begin{aligned}
E[\hat{\tau}_{t+1}|s_t = 1, a_t, k_{t-1}] &= p_{11}\hat{\tau}_1 + (1 - p_{11})\hat{\tau}_2 \\
E[\hat{\tau}_{t+1}|s_t = 2, a_t, k_{t-1}] &= (1 - p_{22})\hat{\tau}_1 + p_{22}\hat{\tau}_2,
\end{aligned}$$

which implies $E_t\hat{\tau}_{t+1} = P\hat{\tau}_t$, where $\hat{\tau}_t = [\hat{\tau}_1 \ \ \hat{\tau}_2]^T$. In general,

$$E_t\hat{\tau}_{t+i} = P^i\hat{\tau}_t \quad \text{and} \quad E_t\hat{\tau}_{t+1+i} = P^{i+1}\hat{\tau}_t.$$

Using (2.8), we obtain

$$\begin{aligned}
k_t &= \alpha k_{t-1} + a_t - \nu_1 \sum_{i=1}^{\infty} [\alpha\beta(1 - \tau)P]^i \hat{\tau}_t \\
&= \alpha k_{t-1} + a_t - \nu_1\alpha\beta(1 - \tau)P \sum_{i=0}^{\infty} [\alpha\beta(1 - \tau)P]^i \hat{\tau}_t \\
&= \alpha k_{t-1} + a_t - \nu_1\alpha\beta(1 - \tau)P[I - \alpha\beta(1 - \tau)P]^{-1}\hat{\tau}_t.
\end{aligned}$$

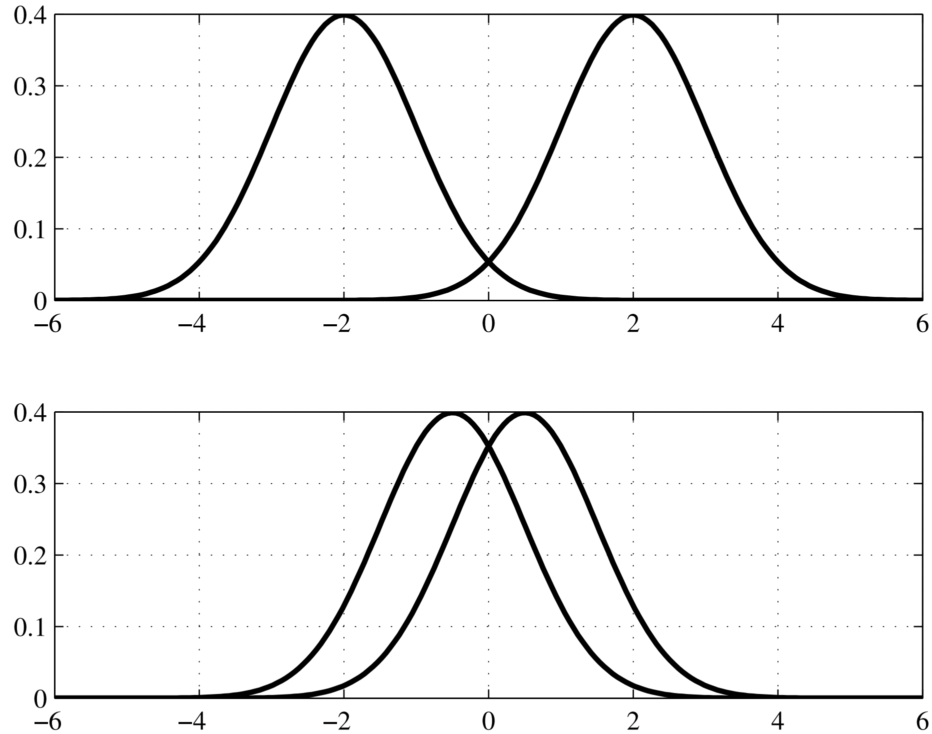

Figure 2.1: Distributions of Taxes

This derivation assumes the tax state, s_t, is observed at time t. Now, suppose that s_t is not observed and the tax process evolves according to

$$\tau_t = \bar{\tau}(s_t) + \varepsilon_t,$$

where $\varepsilon \sim N(0, \sigma^2)$. The household observes τ_t, but given that the state is hidden, does not know whether changes in taxes are due to changes in the intercept or temporary tax shocks. Hence, the household must solve a signal extraction problem. The two cases in figure 2.1 help illustrate the difficulty the household may face when forming inferences about the state. In the top panel, the distributions do not overlap very much. Thus, it will be fairly easy for the household to infer the state. In the bottom panel, the distributions substantially overlap, and most of the time it will be difficult for the household to infer the state.

2.1.4 Lucas Critique

We can draw three important conclusions from (2.8):

1. Only expected taxes matter, unexpected taxes are treated as lump sum
2. More distant expected taxes are discounted more heavily
3. Present value taxes matter, not timing

Consider two cases for the process governing taxes:

Case 1 Suppose $\hat{\tau}_t = \varepsilon_t$, where $\varepsilon_t \sim i.i.d.$ Then $E_t\hat{\tau}_{t+i} = 0$ for all i and (2.8) reduces to

$$k_t = \alpha k_{t-1} + a_t.$$

Case 2 Suppose $\hat{\tau}_t = \rho\hat{\tau}_{t-1} + \varepsilon_t$, where $\varepsilon \sim i.i.d.$ Then $E_t\hat{\tau}_{t+i} = \rho^i\hat{\tau}_t$ for $i > 0$. Then the process for capital, (2.8), becomes

$$\begin{aligned} k_t &= \alpha k_{t-1} + a_t - \nu_1 \sum_{j=1}^{\infty} (\theta\rho)^j \hat{\tau}_t \\ &= \alpha k_{t-1} + a_t - \eta_{k,\tau}\hat{\tau}_t, \end{aligned} \tag{2.9}$$

where $\eta_{k,\tau} \equiv \nu_1\theta\rho/(1-\theta\rho)$ is the elasticity of k with respect to $\hat{\tau}$. Moreover, $\eta_{k,\tau}$ is increasing in ρ. The more persistent the tax process, the more k will respond to $\hat{\tau}$.

These results make plain that if the process governing the evolution of $\hat{\tau}$ changes, then the process governing the equation for k_t changes. Lucas (1976) criticized a range of econometric policy evaluation procedures because they used models that assumed private agents' decision rules are invariant to the laws of motion they faced. Suppose an econometrician estimates some macroeconomic relationship, which does not identify the structural parameters, and uses the results to determine the impact of alternative policies. The *Lucas Critique* says the econometrician's predictions will be incorrect because the structural parameters are *not* policy invariant.

Putting (2.9) and the process for taxes into vector auto-regression (VAR) form, we obtain

$$\begin{aligned} \begin{bmatrix} 1 & \eta_{k,\tau} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_t \\ \hat{\tau}_t \end{bmatrix} &= \begin{bmatrix} \alpha & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_{t-1} \\ \hat{\tau}_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_t \\ \varepsilon_t \end{bmatrix} \\ \rightarrow \begin{bmatrix} k_t \\ \hat{\tau}_t \end{bmatrix} &= \begin{bmatrix} \alpha & -\rho\eta_{k,\tau} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_{t-1} \\ \hat{\tau}_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & -\eta_{k,\tau} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_t \\ \varepsilon_t \end{bmatrix}. \end{aligned} \tag{2.10}$$

Given the process for taxes, for some $\varepsilon_t > 0$, $E_t\hat{\tau}_{t+1}$ increases assuming $\rho \neq 0$. This is because ε_t will have an effect on $\hat{\tau}_t$ and in turn effect $E_t\hat{\tau}_{t+1} = \rho\hat{\tau}_t$. ε_t alters agents' conditional expectations but not agents' expectations functions. Thus, the reduced form VAR, given in (2.10), is structural (invariant to interventions) with respect to ε, but not with respect to ρ.

For example, least squares regression of

$$\begin{aligned} k_t &= b_{11}k_{t-1} + b_{12}\hat{\tau}_{t-1} + \epsilon_{1t} \\ \hat{\tau}_t &= b_{22}\hat{\tau}_{t-1} + \epsilon_{2t} \end{aligned}$$

will provide estimates of $\{b_{11}, b_{12}, b_{22}, \epsilon_{1t}, \epsilon_{2t}\}$. Since $b_{11} = \alpha$, $b_{22} = \rho$, and $\epsilon_{2t} = \varepsilon_t$, it is easy to see that the remaining equations

$$\begin{aligned} b_{12} &= -\rho\eta_{k,\tau}(\alpha, \beta, \rho) \\ \epsilon_{1t} &= a_t - \eta_{k,\tau}(\alpha, \beta, \rho)\varepsilon_t \end{aligned}$$

will precisely identify the parameter β. Therefore, this model is just-identified and permits us to recover the complete set of structural parameters of the economy.

Now suppose the econometrician estimates the reduced form policy function for capital. Since $\eta_{k,\tau}$ is treated as a constant, the forecasts are unaffected by changes in ρ. This is because the econometrician is unable to distinguish in his/her data set between changes in ρ and ε. The problem is that the policy function is *not* invariant to ρ.

2.1.5 Solution Method II: MSV Approach

To focus attention on the technique, assume $\hat{\tau}_t = 0$ for all t. Then (2.7) reduces to

$$E_t k_{t+1} - \left[\frac{1}{\alpha\beta} + \alpha\right] k_t + \frac{1}{\beta} k_{t-1} = -\frac{1}{\alpha\beta} a_t.$$

Instead of solving this difference equation directly, posit a Minimum State Variable (MSV) solution of the form

$$k_t = b_1 k_{t-1} + b_2 a_t,$$

where b_1 and b_2 are unknown coefficients that we will have to pin down. After plugging our guess into the difference equation, we arrive at

$$[b_1 - (\alpha\beta)^{-1} - \alpha][b_1 k_{t-1} + b_2 a_t] + \beta^{-1} k_{t-1} = -(\alpha\beta)^{-1} a_t.$$

Equating the coefficients on the state variables yields

$$[b_1 - \alpha][b_1 - (\alpha\beta)^{-1}] = 0$$
$$[b_1 - (\alpha\beta)^{-1} - \alpha] b_2 = -(\alpha\beta)^{-1}.$$

Using the stable root, it is easy to see that $b_1 = \alpha$ and $b_2 = 1$. Therefore,

$$k_{t+1} = \alpha k_t + a_{t+1},$$

which is the same solution we arrive at using Jordan decomposition. In order to obtain the path of consumption, once again posit an MSV solution given by

$$c_t = d_1 k_{t-1} + d_2 a_t,$$

where d_1 and d_2 are unknown coefficients that we will have to pin down. After plugging in our guess into (2.5), we arrive at

$$d_1 k_{t-1} + d_2 a_t - d_1[\alpha k_{t-1} + a_t] = (1-\alpha)[\alpha k_{t-1} + a_t].$$

After equating coefficients on the state variables, it is easy to see that $d_1 = \alpha$ and $d_2 = 1$. Therefore,

$$c_{t+1} = \alpha k_t + a_{t+1},$$

2.2 Numerical Solution Method

2.2.1 Introduction to Gensys

Gensys is Matlab code written by Chris Sims that is designed to solve stochastic linear rational expectations models. This section provides a brief introduction on how to use the program. The key to using this program is to map the model into the following form:

$$G_0 X_{t+1} = G_1 X_t + \Psi \varepsilon_{t+1} + \Pi \eta_{t+1} + C, \tag{2.11}$$

where X is a vector of variables (exogenous and endogenous), ε is a vector of exogenous random variables (shocks), C is a vector of constants (often zero), and η is a vector of forecast errors whose elements satisfy

$$\eta^x_{t+1} = x_{t+1} - E_t x_{t+1}$$

for some $x \in X$. After mapping the model into the form given in (2.11), `gensys` is called with a statement of the form:

$$[G, C, M, F, A, B, \texttt{gev}, eu] = \texttt{gensys}(G0, G1, C, Psi, Pi, div).$$

The last argument, div, determines the size of the root that is treated as "unstable" for determining existence and uniqueness. By default, if this argument is omitted, the program assumes that any root strictly greater than one is suppressed. You will rarely, if ever, need to specify this argument since most of the time the default value is preferred.

The output of *gensys* is given by

$$X_t = GX_{t-1} + C + M\epsilon_t + \sum_{s=0}^{\infty} AF^s BE_t\epsilon_{t+s+1},$$

where G governs the evolution of the endogenous variables, M is the impact matrix, F is a matrix that discounts the forward solution to the present, and A and B and weighting matrices. When shocks are $i.i.d$ mean zero, the last term is equal to zero.

The returned value eu is a 2×1 vector whose first element characterizes existence of an equilibrium (1 if true, 0 if false) and whose second element characterizes uniqueness of the equilibrium (1 if true, 0 if false). Thus, we hope to obtain $eu = [1 \;\; 1]$. Finally, the returned value gev is the generalized eigenvalue matrix. Its second column divided by its first column is the vector of eigenvalues of $G_0^{-1}G_1$ if G_0 is invertible, and its first column divided by its second column is the vector of eigenvalues of $G_0G_1^{-1}$ if G_1 is invertible.

The simplicity of using this program is illustrated in the following example.

2.2.2 Model Setup

Consider a stochastic Robinson Crusoe economy, where a social planner chooses $\{C_t, N_t, K_t\}_{t=0}^{\infty}$ to maximize lifetime utility, given by,

$$E_0 \sum_{t=0}^{\infty} \beta^t \{\ln C_t + \theta \ln(1 - N_t)\}, \qquad \theta \geq 0, \tag{2.12}$$

subject to

$$C_t + K_t = Y_t + (1 - \delta)K_{t-1}, \tag{2.13}$$

$$Y_t = A_t K_{t-1}^{1-\alpha} N_t^{\alpha}, \tag{2.14}$$

where $0 \leq \alpha \leq 1$. The technology shock follows

$$A_t = \bar{A}(A_{t-1}/\bar{A})^{\rho} \exp(\varepsilon_t), \tag{2.15}$$

where $\bar{A}$ is steady-state technology, $0 \leq \rho \leq 1$, $\varepsilon \sim N(0, \sigma_A^2)$.

The first order conditions of the social planner's problem are given by

$$\frac{1}{C_t} = \beta E_t \left\{ \frac{1}{C_{t+1}} R_{t+1} \right\} \tag{2.16}$$

$$\frac{\theta}{1 - N_t} = \frac{\alpha Y_t}{C_t N_t} \tag{2.17}$$

$$R_t = (1 - \alpha)\frac{Y_t}{K_{t-1}} + (1 - \delta). \tag{2.18}$$

Equations (2.13)-(2.18) form a system of 6 equations and 6 variables, which can be solved numerically using Chris Sims's `gensys.m` program. For further details, see Sims (2002).

2.2.3 Deterministic Steady State

Obtaining steady state values can sometimes be tricky. It is important to work from the simplest steady state equations to the most complicated. In this case, notice that (2.16) implies

$$R = 1/\beta.$$

Using (2.18), we can calculate $K/Y = (1-\alpha)(R+\delta-1)^{-1}$. Thus, using the aggregate resource constraint, $C/Y = 1 - \delta K/Y$. Using the production function,

$$\frac{N}{Y} = \left(\frac{K}{Y}\right)^{(\alpha-1)/\alpha}.$$

Setting $N = 1/3$, which is consistent with a standard work day, from (2.17)

$$\theta = \alpha \left(\frac{C}{Y}\right)^{-1} \left(\frac{1-N}{N}\right).$$

Using N and N/Y we can calculate steady state Y and hence C and K. Assuming the following parameter settings,

$$\alpha = 2/3 \quad \beta = 0.99, \quad \delta = 0.025,$$

straightforward numerical calculations imply

$$(K, N, C, Y) = (10.3, 0.33, 0.81, 1.06).$$

2.2.4 Log-Linear System

Log-linearizing (2.13)-(2.18), we obtain

$$\begin{aligned}
-c_t &= -E_t c_{t+1} + E_t r_{t+1}, \\
\frac{1}{1-N} n_t &= y_t - c_t, \\
C c_t + K k_t &= Y y_t + (1-\delta) K k_{t-1}, \\
y_t &= a_t + (1-\alpha) k_{t-1} + \alpha n_t, \\
R r_t &= (1-\alpha)(Y/K)(y_t - k_{t-1}), \\
a_t &= \rho a_{t-1} + \varepsilon_t,
\end{aligned}$$

where a lower case letter denotes log deviations from the deterministic steady state. Introducing forecast errors, the linearized consumption euler equation can be written as

$$c_{t+1} - r_{t+1} = c_t + \eta^c_{t+1} - \eta^r_{t+1}.$$

2.2.5 Mapping the Model into Gensys Form

Putting the linearized system into matrix form, we obtain

$$G_0 X_{t+1} = G_1 X_t + \Psi \varepsilon_{t+1} + \Pi \eta_{t+1},$$

where

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1/(1-N) & 1 & 0 & 0 \\ C & K & 0 & -Y & 0 & 0 \\ 1 & 0 & -\alpha & 0 & 0 & -1 \\ 0 & 0 & 0 & -(1-\alpha)(Y/K) & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta)K & 0 & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 & 0 & 0 \\ 0 & -(1-\alpha)(Y/K) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}$$

and

$$\Pi = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad X_{t+1} = \begin{bmatrix} c_{t+1} \\ k_{t+1} \\ n_{t+1} \\ y_{t+1} \\ r_{t+1} \\ a_{t+1} \end{bmatrix}.$$

Now that we have setup the appropriate matrices, all that remains is to simply enter the matrices row by row into Matlab.

2.2.6 Gensys and Moving Average Components

Now assume technology evolves (in deviations from steady state) according to

$$a_t = \rho a_{t-1} + \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1}.$$

In this case, the shock at time $t-1$ affects technology at time t. In other words, households receive one period news about future changes in technology. To map this process into `gensys` form, given in (2.11), and avoid using the forward solution, we must create a dummy variable, defined as

$$d_{1,t+1} = \varepsilon_{t+1}. \tag{2.19}$$

Then the technology process can be written as

$$a_{t+1} = \rho a_t + \theta_0 d_{1,t+1} + \theta_1 d_{1,t}.$$

Thus, we have added 1 new variable, which shows up in the process for technology, and one new equation, (2.19).

Now suppose there is a second moving-average component so technology evolves according to

$$a_t = \rho a_{t-1} + \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}.$$

In this case, we need to define two dummy variables, given by,

$$d_{2,t+1} = \varepsilon_{t+1} \qquad \text{and} \qquad d_{1,t+1} = d_{2,t}.$$

Then the technology process can be written as

$$a_{t+1} = \rho a_t + \theta_0 d_{2,t+1} + \theta_1 d_{2,t} + \theta_2 d_{1,t}.$$

In this case, we have added two new variables and two new equations.

This process of introducing dummy variables generalizes for any number of moving-average components. To map the model into `gensys` form and avoid using the forward solution, the user must introduce a new dummy variable for every period of news.

2.2.7 Gensys and Forward/Lag Variables

Now consider the following utility function with internal habit formation:

$$U(C_t, C_{t-1}, 1-N_t) = \frac{(C_t - hC_{t-1})^{1-\gamma}}{1-\gamma},$$

where γ is the constant of relative risk aversion and h is the degree of habit. For simplicity, we will assume labor is inelastically supplied. The Lagragian is then given by

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(C_t - hC_{t-1})^{1-\gamma}}{1-\gamma} + \lambda_t [A_t K_{t-1}^{1-\alpha} - C_t - K_t + (1-\delta)K_{t-1}] \right\}.$$

The first order conditions imply

$$\lambda_t = (C_t - hC_{t-1})^{-\gamma} - \beta h E_t\{(C_{t+1} - hC_t)^{-\gamma}\},$$
$$\lambda_t = \beta E_t\{\lambda_{t+1} R_{t+1}\}.$$

To linearize the system, you can either combine the first order conditions to substitute out λ or you can leave the equations in their current form and linearize around λ. For consistency with the previous setup, we will combine the first order conditions to obtain

$$(C_t - hC_{t-1})^{-\gamma} - \beta h E_t\{(C_{t+1} - hC_t)^{-\gamma}\} = \beta E_t\{[(C_{t+1} - hC_t)^{-\gamma} - \beta h (C_{t+2} - hC_{t+1})^{-\gamma}] R_{t+1}\}.$$

The log-linearized Euler equation is now given by

$$-\gamma[1 + h(1+\beta h)]c_t + \gamma h c_{t-1} = -\gamma[1 + \beta h(1+h)]E_t c_{t+1} + \beta\gamma h E_t c_{t+2} + (1-h)(1-\beta h)E_t r_{t+1}.$$

Notice that if $h = 0$, this equation reduces to the Euler equation derived above without habit formation. In this case, we have a third order difference equation. To map this equation into `gensys` form, we must define lead and lag variables. First define a dummy variable, d_1, such that $d_{1,t+1} = E_t c_{t+2} = c_{t+2} - \eta^{d_1}_{t+2}$. Notice that this implies

$$c_{t+1} = d_{1,t} + \eta^{d_1}_{t+1}. \tag{2.20}$$

Second, define a variable d_2 such that

$$d_{2,t+1} = c_t. \tag{2.21}$$

Then the linearized consumption Euler equation can be written as

$$\begin{aligned} &- \gamma[1 + \beta h(1+h)]c_{t+1} + \beta\gamma h d_{1,t+1} + (1-h)(1-\beta h) r_{t+1} \\ &= -\gamma[1 + h(1+\beta h)]c_t + \gamma h d_{2,t} - \gamma[1 + \beta h(1+h)]\eta^c_{t+1} + \beta\gamma h \eta^{d_1}_{t+1} + (1-h)(1-\beta h)\eta^r_{t+1} \end{aligned}$$

In short, we have added two new variables (d_1 and d_2) and two new equations, (2.20) and (2.21), so that the system conforms with (2.11).

2.2.8 Gensys: Behind the Code

To more clearly understand the techniques applied by `gensys`, recall the conventional growth model that was introduced in section 2.1. Assuming $\tau_t = 0$ for all t and a_t is i.i.d with mean zero, equations (2.5) and (2.6) can be written as

$$-c_t = (\alpha - 1)k_t - E_t c_{t+1}$$
$$c_t = \frac{1}{1-\alpha\beta}a_t + \frac{\alpha}{1-\alpha\beta}k_{t-1} - \frac{\alpha\beta}{1-\alpha\beta}k_t.$$

Defining the forecast error as $\eta_{t+1} = c_{t+1} - E_t c_{t+1}$, the above system is given by

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & \frac{\alpha\beta}{1-\alpha\beta} \end{bmatrix}}_{G_0} \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \alpha-1 \\ 0 & \frac{\alpha}{1-\alpha\beta} \end{bmatrix}}_{G_1} \begin{bmatrix} c_t \\ k_t \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\Pi} \eta_{t+1} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{1-\alpha\beta} \end{bmatrix}}_{\Psi} a_{t+1}.$$

After inverting the G_0 matrix, we obtain

$$\begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \alpha-1 \\ \frac{\alpha\beta-1}{\alpha\beta} & \frac{1-\alpha\beta(1-\alpha)}{\alpha\beta} \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{\alpha\beta-1}{\alpha\beta} \end{bmatrix} \eta_{t+1} + \begin{bmatrix} 0 \\ \frac{1}{\alpha\beta} \end{bmatrix} a_{t+1},$$

which can be written more compactly as

$$x_{t+1} = Ax_t + B\xi_{t+1}, \tag{2.22}$$

where A is the coefficient matrix in the above system and

$$x_t = (c_t, k_t)' \qquad \xi_t = (a_t, \eta_t)' \quad B = \begin{bmatrix} 0 & 1 \\ \frac{1}{\alpha\beta} & \frac{\alpha\beta-1}{\alpha\beta} \end{bmatrix}.$$

Solving for the zeros of the characteristic equation, $\det(A - \lambda I) = 0$, implies that the eigenvalues are given by $\lambda_1 = \alpha < 1$ and $\lambda_2 = (\alpha\beta)^{-1}$. To solve for the corresponding eigenvectors, $\mathbf{v}_i = [v_{1i} \;\; v_{2i}]$, evaluate the equation $A\mathbf{v}_i = \lambda\mathbf{v}_i$ for each eigenvalue $i = 1, 2$. That is for $\lambda = \alpha$

$$A - \alpha I = \begin{bmatrix} 1-\alpha & \alpha-1 \\ \frac{\alpha\beta-1}{\alpha\beta} & \frac{1-\alpha\beta(1-\alpha)}{\alpha\beta} - \alpha \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha-1 \\ \frac{\alpha\beta-1}{\alpha\beta} & \frac{1-\alpha\beta}{\alpha\beta} \end{bmatrix} \overset{rref}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus, normalizing $v_{12} = 1$ implies that the eigenvector corresponding to λ_1 is $\mathbf{v}_1 = [1 \;\; 1]'$. For $\lambda = (\alpha\beta)^{-1}$

$$A - (\alpha\beta)^{-1}I = \begin{bmatrix} 1-(\alpha\beta)^{-1} & \alpha-1 \\ \frac{\alpha\beta-1}{\alpha\beta} & \frac{1-\alpha\beta(1-\alpha)}{\alpha\beta} - \frac{1}{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \frac{\alpha\beta-1}{\alpha\beta} & \alpha-1 \\ \frac{\alpha\beta-1}{\alpha\beta} & \alpha-1 \end{bmatrix} \overset{rref}{\sim} \begin{bmatrix} 1 & \frac{\alpha\beta(\alpha-1)}{\alpha\beta-1} \\ 0 & 0 \end{bmatrix}.$$

Thus, normalizing $v_{22} = 1$ implies that the eigenvector corresponding to λ_2 is $\mathbf{v}_2 = [\alpha\beta(\alpha-1)/(1-\alpha\beta) \;\; 1]'$. Making use of the Jordan decomposition of the coefficient matrix A, $A = PDP^{-1}$ and we will denote $P_{\cdot j}$ as the j^{th} column of matrix P and $P^{j\cdot}$ as the j^{th} row of matrix P^{-1}.[1] Given the above results,

$$P = \begin{bmatrix} 1 & \frac{\alpha\beta(\alpha-1)}{1-\alpha\beta} \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{1-\alpha\beta}{1-\alpha^2\beta} & \frac{\alpha\beta(1-\alpha)}{1-\alpha^2\beta} \\ \frac{\alpha\beta-1}{\alpha^2\beta-1} & \frac{1-\alpha\beta}{1-\alpha^2\beta} \end{bmatrix}.$$

[1] Note that if $A = PDP^{-1}$, then $A^2 = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$. Hence, $A^j = PD^jP^{-1}$.

Iterating (2.22) backwards yields

$$x_t = A^t x_0 + \sum_{s=0}^{t-1} A^s B \xi_{t-s}.$$

Applying the Jordan Decomposition, we obtain

$$x_t = \sum_{j=1}^{2} P_{\cdot j} \lambda_j^t P^{j\cdot} x_0 + \sum_{j=1}^{2} P_{\cdot j} \sum_{s=0}^{t-1} \lambda_j^s P^{j\cdot} B \xi_{t-s}$$

To eliminate the influence of explosive eigenvalues ($|\lambda_j| > 1$), we need to impose that for each explosive eigenvalue

$$P^{j\cdot} x_t = 0, \qquad t = 0, 1, 2, \ldots$$

or, equivalently,

$$P^{j\cdot} x_0 = 0.$$

As well as

$$P^{j\cdot} B \xi_t = 0 \qquad t = 0, 1, 2, \ldots$$

The first condition places the solution onto the stable manifold, while the second condition ensures that exogenous shocks affect endogenous forecast errors in a way that keeps the solution on the stable manifold.

In this model, $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Thus, we need to squash the influence of $\lambda_2 = \frac{1}{\alpha\beta}$ by setting

$$P^{2\cdot} x_t = \begin{bmatrix} \frac{\alpha\beta-1}{\alpha^2\beta-1} & \frac{1-\alpha\beta}{1-\alpha^2\beta} \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} = 0,$$

which implies that $c_t = k_t$. Also,

$$P^{2\cdot} B \xi_t = \begin{bmatrix} \frac{\alpha\beta-1}{\alpha^2\beta-1} & \frac{1-\alpha\beta}{1-\alpha^2\beta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{\alpha\beta} & \frac{\alpha\beta-1}{\alpha\beta} \end{bmatrix} \begin{bmatrix} a_t \\ \eta_t \end{bmatrix} = 0,$$

implying that $\eta_t = a_t$. Given the above result, we know $c_{t+1} - k_{t+1} = 0$ and $\eta_{t+1} = a_{t+1}$. Thus, the system can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \alpha - 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} a_{t+1} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} a_{t+1},$$

which yields the solution

$$\begin{aligned} k_{t+1} &= \alpha k_t + a_{t+1} \\ c_{t+1} &= \alpha k_t + a_{t+1}. \end{aligned}$$

Gensys will return the solution in the form $x_{t+1} = G x_t + M a_{t+1}$, where

$$G = \begin{bmatrix} 0 & \alpha \\ 0 & \alpha \end{bmatrix} \qquad M = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which is the same solution we arrived at using the direct approach (section 2.1.3) and the MSV approach (section 2.1.5).

Chapter 3

Real Business Cycle Models

3.1 Basic Facts about Economic Fluctuations

Real business cycles are recurrent fluctuations of output about trend and the co-movements of other aggregate time series. The following are some of the key facts:

- Average output growth is roughly constant and fluctuations in output growth are distributed roughly symmetrically around its mean. That is, there is a a constant trend in output growth and there are no large asymmetries between rises and falls in output growth.

- The average growth rate in capital is roughly constant and approximately equal to the average growth rate of output. This implies that the capital to output ratio is constant.

- Labor and capital receive constant shares of total income. Given that the capital-to-output ratio is approximately constant, this implies a constant return on capital. The labor share is the fraction of output that goes to workers in the form of wages. Similarly, the capital share measures aggregate payments to capital divided by output.

- Business cycles do not exhibit any simple regular or cyclical pattern. Thus, the prevailing view is that the economy is perturbed by disturbances of various types and sizes at more or less random intervals, and that those disturbances then propagate through the economy. Where the major macroeconomic schools of thought differ is in their hypotheses concerning these shocks and propagation mechanisms.

- Business cycles are distributed very unevenly over the components of output. The average share of each of the components in total output is often very different from its share in the declines in output (relative to its normal growth) in recession. For example, investment accounts for 16 percent of output but 75 percent of the shortfall in growth relative to normal in recessions.

3.2 Campbell: Inspecting the Mechanism

The stochastic growth model is the workhorse model in macroeconomic analysis. There is an emphasis on technology shocks as the source of business cycle dynamics, but many other shocks have also been studied (e.g. government spending, taxes). Campbell (1994) analytically solves the stochastic growth model with government spending and technology shocks using the minimum state variable (MSV) approach. This paper does an excellent job explaining the basic mechanisms driving the model solution. The author primarily focuses on technology shocks using two models, one with a fixed labor supply and one with a variable labor supply.

3.2.1 Model 1: Fixed Labor Supply

Model Setup

The social planner chooses $\{C_t, K_t\}_{t=0}^{\infty}$ to maximize lifetime utility, given by,

$$E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}) = E_t \sum_{i=0}^{\infty} \beta^i \frac{C_{t+i}^{1-\gamma}}{1-\gamma},$$

where γ is the coefficient of relative risk aversion and $\sigma = 1/\gamma$ is the elasticity of intertemporal substitution. A large σ implies a willingness to substitute across time, which leads to more volatile consumption. E_t in the expectations operator conditional on the information set at t, which includes all variables dated t and earlier. These choices are constrained by

$$C_t + K_t = (A_t N_t)^{\alpha} K_{t-1}^{1-\alpha} + (1-\delta)K_{t-1} = Y_t + (1-\delta)K_{t-1}, \tag{3.1}$$

where $Y_t = F(K_{t-1}, A_t N_t) \equiv (A_t N_t)^{\alpha} K_{t-1}^{1-\alpha}$ denotes production.

The state of the economy at time t is (A_t, K_{t-1}). Thus, decisions at time t are a function of (A_t, K_{t-1}) and nothing else. In equilibrium $N_t = 1$ for all t and the Euler equation is given by

$$1 = \beta E_t \left\{ (C_{t+1}/C_t)^{-\gamma} R_{t+1} \right\}, \tag{3.2}$$

where R_{t+1} is the gross rate of return on a one-period investment in capital and is given by

$$R_{t+1} = \underbrace{(1-\alpha)\left(A_{t+1}/K_t\right)^{\alpha}}_{\text{Marginal Product of Capital}} + \underbrace{(1-\delta)}_{\text{Undepreciated Capital}} \tag{3.3}$$

Balanced Growth Path

In steady state, technology grows at a constant rate $G \equiv A_{t+1}/A_t$, which is exogenous. Along the balanced growth path C, K, and Y all grow at the same rate, meaning

$$G = \frac{Y_{t+1}}{Y_t}, \qquad G = \frac{K_{t+1}}{K_t}, \qquad G = \frac{C_{t+1}}{C_t}.$$

From the Euler equation, (3.2), in steady state we know

$$1 = \beta G^{-\gamma} R \qquad \to \qquad G^{\gamma} = \beta R$$

Hence, along the balanced growth path, the real interest rate is constant and given by

$$R_{t+1} = \bar{R} = (1-\alpha)(\bar{A}/\bar{K}) + (1-\delta).$$

If we define $\ln G = g$ and $\ln R = r$, then

$$g = \sigma(\ln \beta + r). \tag{3.4}$$

We can rewrite the real return, (3.3), as

$$\frac{A_{t+1}}{K_t} = \left(\frac{R_{t+1} - (1-\delta)}{(1-\alpha)} \right)^{1/\alpha}.$$

Hence, since $R = e^r \approx 1 + r$, in steady state, we obtain

$$\begin{aligned}\frac{\bar{A}}{\bar{K}} &= \left(\frac{\bar{R} - (1-\delta)}{(1-\alpha)}\right)^{1/\alpha} = \left(\frac{G^\gamma/\beta - (1-\delta)}{(1-\alpha)}\right)^{1/\alpha} \\ &\approx \left(\frac{(1+r) - (1-\delta)}{(1-\alpha)}\right)^{1/\alpha} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha}. \end{aligned} \tag{3.5}$$

A higher growth rate of technology leads to a lower level of capital per unit of technology, because faster technology growth must be accompanied by higher consumption growth. Agents accept a steeper consumption path only if the rate of return on capital is higher, which implies a lower level of capital. Express the balanced growth path in terms of (g, r, α, δ). Calibrate these parameters to the following benchmark values:

- $g = 0.005$ (2% annual growth rate)
- $r = 0.015$ (6% annual rate of return)
- $\alpha = 0.666$ (Labor Share)
- $\delta = 0.025$ (10% annual depreciation rate)

Use (3.4) to get (σ, β) pairs that are consistent with these parameters, since they are not separately identified by long-run averages.

The production function can be written as

$$\frac{Y_t}{K_{t-1}} = \left(\frac{A_t}{K_{t-1}}\right)^\alpha.$$

Hence, along the balanced growth path

$$\frac{\bar{Y}}{\bar{K}} = \left(\frac{\bar{A}}{\bar{K}}\right)^\alpha \approx \left(\frac{r+\delta}{1-\alpha}\right). \tag{3.6}$$

Dividing the budget constraint, (3.1), by K_{t-1}, we obtain

$$\frac{C_t}{K_{t-1}} = (1-\delta) + \left(\frac{A_t}{K_{t-1}}\right)^\alpha - \frac{K_t}{K_{t-1}}$$

Therefore, along the balanced growth path

$$\begin{aligned}\frac{\bar{C}}{\bar{K}} &= (1-\delta) + \left(\frac{\bar{A}}{\bar{K}}\right)^\alpha - G \\ &\approx (1-\delta) + \left(\frac{r+\delta}{1-\alpha}\right) - (1+g)\end{aligned}$$

and

$$\frac{\bar{C}}{\bar{Y}} = \frac{\bar{C}/\bar{K}}{\bar{Y}/\bar{K}} = 1 - \frac{(1-\alpha)(g+\delta)}{r+\delta}. \tag{3.7}$$

The benchmark parameters, given above, imply

$$\begin{aligned}\frac{\bar{Y}}{\bar{K}} &= 0.118 \quad \text{(0.47\% annual rate)} \\ \frac{\bar{C}}{\bar{Y}} &= 0.745.\end{aligned}$$

Log-Linear System

An exact analytical solution is only possible in the unrealistic case where capital depreciates fully in one period and where agents have log utility in consumption. Thus, we seek an approximate solution by transforming the model into a system of log linear difference equations. The variables in the system can be thought of as zero-mean deviations from the steady state growth path.

Linearizing the production function, we obtain

$$y_t = \alpha a_t + (1-\alpha)k_{t-1}. \tag{3.8}$$

To linearize the resource constraint, (3.1), we must first make the equation stationary by dividing thru by K_{t-1}. After dividing, we obtain

$$\frac{K_t}{K_{t-1}} = (1-\delta) + \frac{Y_t}{K_{t-1}} - \frac{C_t}{K_{t-1}}.$$

Linearizing this result yields

$$(1+g)(k_t - k_{t-1}) = (\bar{Y}/\bar{K})(y_t - k_{t-1}) - (\bar{C}/\bar{K})(c_t - k_{t-1})$$

Solving for k_t and substituting for the steady state shares using (3.6) and (3.7) implies

$$\begin{aligned}
k_t &= \frac{\bar{Y}}{\bar{K}}\frac{1}{1+g}y_t - \frac{\bar{Y}}{\bar{K}}\frac{1}{1+g}k_{t-1} - \frac{\bar{C}}{\bar{K}}\frac{1}{1+g}c_t + \frac{\bar{C}}{\bar{K}}\frac{1}{1+g}k_{t-1} + k_{t-1}\\
&= \left[1 - \frac{\bar{Y}}{\bar{K}}\frac{1}{1+g}\left(1-\frac{\bar{C}}{\bar{Y}}\right)\right]k_{t-1} + \frac{\bar{Y}}{\bar{K}}\frac{1}{1+g}y_t - \frac{\bar{C}}{\bar{K}}\frac{1}{1+g}c_t\\
&= \left[1 - \frac{r+\delta}{1-\alpha}\frac{1}{1+g}\frac{(1-\alpha)(g+\delta)}{r+\delta}\right]k_{t-1} + \frac{\bar{Y}}{\bar{K}}\frac{1}{1+g}y_t - \frac{\bar{C}}{\bar{K}}\frac{1}{1+g}c_t\\
&= \frac{1-\delta}{1+g}k_{t-1} + \left(\frac{r+\delta}{1-\alpha}\frac{1}{1+g}\right)(\alpha a_t + (1-\alpha)k_{t-1}) - \frac{\bar{C}}{\bar{K}}\frac{1}{1+g}c_t\\
&= \frac{1+r}{1+g}k_{t-1} + \frac{\alpha}{1-\alpha}\frac{r+\delta}{1+g}a_t + \left[1 - \frac{1-\delta}{1+g} - \frac{r+\delta}{1-\alpha}\frac{1}{1+g} - \frac{r+\delta}{1+g} + \frac{r+\delta}{1+g}\frac{1-\alpha}{1-\alpha}\right]c_t\\
&= \frac{1+r}{1+g}k_{t-1} + \frac{\alpha}{1-\alpha}\frac{r+\delta}{1+g}a_t + \left[1 - \frac{1+r}{1+g} - \frac{\alpha}{1-\alpha}\frac{r+\delta}{1+g}\right]c_t.
\end{aligned}$$

This implies

$$k_t = \lambda_1 k_{t-1} + \lambda_2 a_t + (1-\lambda_1-\lambda_2)c_t, \tag{3.9}$$

where

$$\lambda_1 = \frac{1+r}{1+g}, \qquad \lambda_2 = \frac{\alpha}{1-\alpha}\frac{r+\delta}{1+g}.$$

Note that if $g > r$, then the above system is dynamically inefficient. In this case, the present value of future income would be infinite. Thus, in this neoclassical growth model, it must be true that $g < r$ ($\lambda_1 > 1$).

Linearizing (3.2), we obtain

$$\beta R(-\gamma)G^{-\gamma}(E_t c_{t+1} - c_t) + \beta G^{-\gamma} R E_t r_{t+1} = 0.$$

Thus, after rearranging and imposing steady state, we obtain

$$E_t r_{t+1} = \gamma(E_t c_{t+1} - c_t). \tag{3.10}$$

Linearizing (3.3) and substituting for $\bar{A}/\bar{K}$ using (3.5) yields

$$\begin{aligned}\bar{R} r_{t+1} &= \alpha(1-\alpha)\left(\frac{\bar{A}}{\bar{K}}\right)^{\alpha}(a_{t+1} - k_t)\\ &= \alpha(1-\alpha)\left(\frac{r+\delta}{1-\alpha}\right)(a_{t+1} - k_t)\\ &= \alpha(r+\delta)(a_{t+1} - k_t).\end{aligned}$$

Since $R = e^r \approx 1 + r$, we obtain

$$r_{t+1} = \lambda_3(a_{t+1} - k_t), \tag{3.11}$$

where $\lambda_3 = \alpha(r+\delta)/(1+r)$. It is important to highlight that λ_3 tends to be small because capital depreciates slowly, thus making changes in technology have small proportional effects on the return to capital (i.e., most of the return is undepreciated capital rather than marginal output).

After combining (3.10) and (3.11), we obtain

$$E_t[\Delta c_{t+1}] = \lambda_3 \sigma E_t[a_{t+1} - k_t], \tag{3.12}$$

where Δ denotes a first difference. Define the technology process, in deviations from the balanced growth path, as

$$a_t = \varphi a_{t-1} + \varepsilon_t, \tag{3.13}$$

where $\varphi \in (-1, 1)$. Equations (3.9), (3.12), and (3.13) define the equilibrium system.

Model Solution

To solve this model we will utilize the minimum state variable approach, which posits linear decision rules as a function of the state, (a_t, k_{t-1}). Define η_{yx} as the partial elasticity of y with respect to x and guess

$$c_t = \eta_{ck} k_{t-1} + \eta_{ca} a_t \tag{3.14}$$
$$k_t = \eta_{kk} k_{t-1} + \eta_{ka} a_t, \tag{3.15}$$

where (this can be seen by plugging the guess for c_t into (3.9))

$$\begin{aligned}\eta_{kk} &= \lambda_1 + (1 - \lambda_1 - \lambda_2)\eta_{ck}\\ \eta_{ka} &= \lambda_2 + (1 - \lambda_1 - \lambda_2)\eta_{ca}.\end{aligned} \tag{3.16}$$

First plug the guess into the Euler equation, (3.12), to obtain

$$(\eta_{ck} + \sigma\lambda_3)k_t - \eta_{ck} k_{t-1} + (\eta_{ca} - \sigma\lambda_3)E_t a_{t+1} - \eta_{ca} a_t = 0.$$

Now use technology process to substitute out $E_t a_{t+1}$ and the guess for k_t to obtain

$$(\eta_{ck} + \sigma\lambda_3)(\eta_{kk} k_{t-1} + \eta_{ka} a_t) - \eta_{ck} k_{t-1} + [(\eta_{ca} - \sigma\lambda_3)\varphi - \eta_{ca}]a_t = 0.$$

Equate coefficients on k_{t-1} to obtain

$$\begin{aligned} &\eta_{ck}[\lambda_1 + (1-\lambda_1-\lambda_2)\eta_{ck} - 1] = -\sigma\lambda_3[\lambda_1 + (1-\lambda_1-\lambda_2)\eta_{ck}] \\ \rightarrow\ &(1-\lambda_1-\lambda_2)\eta_{ck}^2 + [(\lambda_1 - 1) + \sigma\lambda_3(1-\lambda_1-\lambda_2)]\,\eta_{ck} + \sigma\lambda_3\lambda_1 = 0. \end{aligned}$$

Thus, we have

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0, \tag{3.17}$$

where

$$Q_2 = 1-\lambda_1-\lambda_2, \qquad Q_1 = (\lambda_1 - 1) + \sigma\lambda_3(1-\lambda_1-\lambda_2), \qquad Q_0 = \sigma\lambda_3\lambda_1.$$

The quadratic formula gives two solutions for η_{ck}. With the benchmark set of parameters, one of them is positive ($\lambda_1 > 1, \lambda_2 > 0, \lambda_3 > 0$, which implies $Q_0 > 0$, $Q_1 > 0$, $Q_2 < 0$). By (3.9), stability requires that $\lambda_1 > 1$, which implies $\eta_{ck} > 0$ if the steady state is locally stable. To see this, suppose $\eta_{ck} < 0$. Then if k_{t-1} rises, we know c_t fall. Therefore k_t rises, which again implies c_{t+1} falls and k_{t+1} rises. This process implies $k \to \infty$. Thus, choose the positive solution to (3.17), which implies

$$\eta_{ck} = \frac{1}{2Q_2}\left\{-Q_1 - \sqrt{Q_1^2 - 4Q_2Q_0}\right\}$$

Note that η_{ck} depends only on σ, λ_1, λ_2, λ_3, which are all invariant with respect to φ. Therefore, η_{ck} is invariant with respect to φ.

Now equate coefficients on a_t to obtain

$$\begin{aligned} &(\eta_{ck} + \sigma\lambda_3)\eta_{ka} + (\eta_{ca} - \sigma\lambda_3)\varphi - \eta_{ca} \\ \rightarrow\ &(\eta_{ck} + \sigma\lambda_3)\lambda_2 - \sigma\lambda_3\varphi + [(\sigma\lambda_3 + \eta_{ck})(1-\lambda_1-\lambda_2) + \varphi - 1]\eta_{ca} = 0, \end{aligned}$$

which implies

$$\eta_{ca} = \frac{\eta_{ck}\lambda_2 + \sigma\lambda_3(\varphi - \lambda_2)}{(\varphi - 1) + (1-\lambda_1-\lambda_2)(\eta_{ck} + \sigma\lambda_3)}.$$

Given that we know η_{ca} and η_{ck}, we also know η_{ka} and η_{kk} given the correspondence given in (3.16).

Time Series Properties

By (3.15), we know

$$(1-\eta_{kk}L)k_t = \eta_{ka}a_t \qquad \rightarrow \qquad k_t = \frac{\eta_{ka}}{1-\eta_{kk}L}a_t$$

The exogenous process for technology, given in (3.13), implies

$$(1-\varphi L)a_t = \varepsilon_t \qquad \rightarrow \qquad a_t = \frac{1}{1-\varphi L}\varepsilon_t.$$

Hence, capital follows an AR(2) process, given by,

$$k_t = \frac{\eta_{ka}}{(1-\eta_{kk}L)(1-\varphi L)}\varepsilon_t.$$

Note that the autoregressive coefficients of the k_t process are $0 < \eta_{kk}, \varphi \leq 1$, which are both real. Therefore, this model can not produce oscillating impulse responses.

The production function, given in (3.8), implies

$$\begin{aligned} y_t &= (1-\alpha)Lk_t + \alpha a_t \\ &= \underbrace{\frac{(1-\alpha)\eta_{ka}L}{(1-\eta_{kk}L)(1-\varphi L)}\varepsilon_t}_{\text{Capital Accumulation Effect}} + \underbrace{\frac{\alpha}{1-\varphi L}\varepsilon_t}_{\text{Direct Effect}} \\ &= \frac{\alpha + [(1-\alpha)\eta_{ka} - \alpha\eta_{kk}]L}{(1-\eta_{kk}L)(1-\varphi L)}\varepsilon_t. \end{aligned}$$

Thus, output follows an ARMA(2,1) process.

The policy function for consumption, given in (3.14), implies

$$\begin{aligned} c_t &= \frac{\eta_{ka}L}{(1-\eta_{kk}L)(1-\varphi L)}\eta_{ck}\varepsilon_t + \frac{1}{1-\varphi L}\eta_{ca}\varepsilon_t \\ &= \frac{\eta_{ca} + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})L}{(1-\eta_{kk}L)(1-\varphi L)}\varepsilon_t \end{aligned}$$

Thus, consumption also follows an ARMA(2,1) process and k, c, and y all have the same autoregressive coefficients.

Key Points

Before moving to the model with a variable labor supply, there are several important characteristics that are important to point out:

- The coefficient η_{ck} does not depend on the persistence of the technology shocks, φ, but is increasing in the elasticity of intertemporal substitution, σ. An increase in capital creates a positive income effect (fixing prices, higher capital implies higher income). It also lowers the real interest rate (lower marginal product), which lowers the cost of current consumption and creates a positive substitution effect that is increasing in σ. A higher σ means households are more willing to intertemporally substitute consumption goods, which makes the substitution effect much larger that the income effect. Since $\eta_{kk} = \lambda_1 + Q_2\eta_{ck}$ and $Q_2 < 0$, η_{kk} is decreasing in σ.

- The coefficient η_{ca} is increasing in φ for low values of σ, but is decreasing in φ for higher values of σ. A positive technology shock produces a positive income effect (higher a, increases y), which is increasing in φ (a higher φ implies that the shock has a larger effect on future income). However, a positive technology shock also produces a negative substitution effect, which increases in φ. For $\varphi > 0$, the real interest rate rises (higher marginal product of capital), which deters current consumption. For low values of σ and/or φ, the substitution effects are weak and the income effect dominates. In fact, for $\varphi = 0$, the technology shock is treated as a windfall gain, and there is no substitution effect. For sufficiently high values of σ and φ, the substitution effect dominates and $\eta_{ca} < 0$. Since $\eta_{ka} = \lambda_2 + Q_2\eta_{ca}$ and $Q_2 < 0$, η_{ka} increases whenever η_{ca} falls.

- Changes in σ impact the responses of output to a positive technology shock, which increase with φ. A higher σ initially implies higher output, but lower (yet still positive) output in the long-run. With a high enough σ, households accumulate capital very rapidly (η_{ka} rises for σ and φ high) and then de-accumulate as the shock disappears.

- Capital Accumulation is important for dynamics only when technology shocks are persistent. There is a weak internal propagation mechanism in the model so most of the dynamics in endogenous variables are inherited from the dynamics of the exogenous technology shock (i.e. Under most calibrations, a technology shock does not generate sufficient capital accumulation to have an important effect on output).
- Capital accumulation does not generate short- or long-run "multipliers" in the sense that the response of output to technology is larger than the underlying shock itself. Slower than normal technology implies slower than normal output but not actual declines in output.

3.2.2 Model 2: Variable Labor Supply

There are two key margins to consider when studying the labor market:

- Extensive Margin: Number of workers that are employed. For example, hiring an additional worker increases the extensive margin.
- Intensive Margin: Amount of use extracted within a given extensive margin. For example, increasing the number of hours per worker would increase the intensive margin.

The following are some of the key facts about the labor market:

- The magnitude of fluctuations in output and aggregate hours of work are nearly identical. It is well-known that the business cycle is most clearly manifested in the labor market and this observation is confirmation.
- Employment fluctuates almost as much as output and total hours worked, while average weekly hours fluctuate considerably less. This suggests that most fluctuations in total hours represent movements into and out of the workforce (extensive margin) rather than adjustments in average hours of work (intensive margin). This means that unemployment is an important feature of the business cycle.
- Productivity is slightly procyclical but varies considerably less than output. Procyclicality suggests that firms hoard labor, which guarantees that employee talent will be available when output growth resumes. That is, firms opt to incur labor during recessions, which makes labor productivity appear procyclical.

Model Setup

Assume the per-period utility function is given by

$$U(C_t, 1-N_t) = \log C_t + \frac{\theta(1-N_t)^{1-\gamma_N}}{1-\gamma_N}, \tag{3.18}$$

so that leisure is additively separable from consumption. King et al. (1988) show that if consumption and leisure are additively separable, then log preferences over consumption are necessary to obtain a constant steady state labor supply. To see this, assume the social planner chooses sequences $\{C_t, N_t, K_t\}_{t=0}^{\infty}$ to maximize lifetime utility, given by,

$$\sum_{t=0}^{\infty} \beta^t U(C_t, 1-N_t),$$

subject to the resource constraint, (3.1). The household's optimality conditions imply

$$U_{C_t}(C_t, 1-N_t) = \beta E_t U_{C_{t+1}}(C_{t+1}, 1-N_{t+1})R_{t+1},$$
$$U_{N_t}(C_t, 1-N_t) = U_{C_t}(C_t, 1-N_t)F_{N_t}(K_{t-1}, A_tN_t),$$

where the return on investment and marginal product of labor are

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_t}\right)^{\alpha} + (1-\delta),$$
$$F_{N_t}(K_{t-1}, A_tN_t) = \alpha\left(\frac{A_t}{K_{t-1}}\right)^{\alpha} K_{t-1}N_t^{\alpha-1}.$$

Thus, we need preferences such that

$$\frac{U_{C_{t+1}}(C_{t+1}, 1-N_{t+1})}{U_{C_t}(C_t, 1-N_t)} \qquad \text{and} \qquad \frac{U_{C_t}(C_t, 1-N_t)}{U_{N_t}(C_t, 1-N_t)}K_{t-1}$$

are both constant along the steady state growth path. The condition must hold so that the real interest rate is constant in steady state. The second condition is required since hours worked cannot grow in steady state, since the time devoted to work is bounded by the endowment.

As an empirical matter, there are certain "great ratios" that are fairly constant over time, such as A/K, Y/K, C/K, and I/K. Thus, A, K, Y, C, and I all grow at some rate. However, N does not grow. The following are examples of alternative momentary utility functions:

Example 1 $U(C, 1-N) = \ln C + V(N)$

$$\frac{U_{C_{t+1}}(C_{t+1}, 1-N_{t+1})}{U_{C_t}(C_t, 1-N_t)} = \frac{C_t}{C_{t+1}}$$
$$\frac{U_{C_t}(C_t, 1-N_t)}{U_{N_t}(C_t, 1-N_t)}K_{t-1} = \frac{K_{t-1}}{C_tV'(N_t)}$$

Both of these equations are constant along the steady state growth path.

Example 2 $U(C, 1-N) = \frac{C^{1-\gamma}}{1-\gamma} + V(N)$

$$\frac{U_{C_{t+1}}(C_{t+1}, 1-N_{t+1})}{U_{C_t}(C_t, 1-N_t)} = \left(\frac{C_t}{C_{t+1}}\right)^{\gamma}$$
$$\frac{U_{C_t}(C_t, 1-N_t)}{U_{N_t}(C_t, 1-N_t)}K_{t-1} = \frac{K_{t-1}}{C_t^{\gamma}V'(N_t)}$$

The first equation is constant along the steady state growth path. However, the second equation is not constant along the steady state growth path unless $\gamma = 1$.

Example 3 $U(C, 1-N) = \frac{\left[C^{\rho}(1-N)^{1-\rho}\right]^{1-\gamma}}{1-\gamma}$

$$\frac{U_{C_{t+1}}(C_{t+1}, 1-N_{t+1})}{U_{C_t}(C_t, 1-N_t)} = \frac{\left[C_{t+1}^{\rho}(1-N_{t+1})^{1-\rho}\right]^{-\gamma}(1-N_{t+1})^{1-\rho}\rho C_{t+1}^{\rho-1}}{\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{-\gamma}(1-N_t)^{1-\rho}\rho C_t^{\rho-1}}$$
$$\frac{U_{C_t}(C_t, 1-N_t)}{U_{N_t}(C_t, 1-N_t)}K_{t-1} = -\frac{\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{-\gamma}(1-N_t)^{1-\rho}\rho C_t^{\rho-1}K_{t-1}}{\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{-\gamma}(1-\rho)(1-N_t)^{-\rho}C_t^{\rho}} = \frac{\rho}{1-\rho}\frac{(1-N_t)K_{t-1}}{C_t}$$

Both of the above equations are clearly constant along the steady state growth path.

Example 4 $U(C, 1-N) = \frac{[C^\rho + \theta(1-N)^\rho]^{(1-\gamma)/\rho}}{1-\gamma}$

$$\frac{U_{C_{t+1}}(C_{t+1}, 1-N_{t+1})}{U_{C_t}(C_t, 1-N_t)} = \frac{\left[C_{t+1}^\rho + \theta(1-N_{t+1})^\rho\right]^{(1-\gamma)/\rho} C_{t+1}^{\rho-1}}{\left[C_t^\rho + \theta(1-N_t)^\rho\right]^{(1-\gamma)/\rho} C_t^{\rho-1}}$$

$$\frac{U_{C_t}(C_t, 1-N_t)}{U_{N_t}(C_t, 1-N_t)} K_{t-1} = -\frac{\left[C_t^\rho + \theta(1-N_t)^\rho\right]^{(1-\gamma)/\rho} C_t^{\rho-1} K_{t-1}}{\left[C_t^\rho + \theta(1-N_t)^\rho\right]^{(1-\gamma)/\rho} \theta(1-N_t)^{\rho-1}} = -\frac{C_t^{\rho-1} K_{t-1}}{\theta(1-N_t)^{\rho-1}}$$

Neither equation is constant along the steady state growth path.

The representative household maximizes (3.18) subject to the resource constraint, given in (3.1). The optimality conditions, given in (3.2) and (3.3), remain the same, except that labor supply is no longer unity in equilibrium. Thus, the gross return on investment is given by

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_t}\right)^\alpha + (1-\delta). \tag{3.19}$$

There is also a static condition for the optimal choice of labor given by

$$\theta(1-N_t)^{-\gamma_N} = \alpha\frac{A_t^\alpha}{C_t}\left(\frac{K_{t-1}}{N_t}\right)^{1-\alpha} = \frac{W_t}{C_t}, \tag{3.20}$$

where W_t represents the marginal product of labor.

Balanced Growth Path

Rearranging (3.19), we obtain

$$\frac{A_{t+1}N_{t+1}}{K_t} = \left(\frac{R_{t+1} - (1-\delta)}{1-\alpha}\right)^{1/\alpha}.$$

Following the same techniques as in section 3.2.1, along the balanced growth path,

$$\frac{\bar{A}\bar{N}}{\bar{K}} = \left(\frac{G^\gamma/\beta - (1-\delta)}{(1-\alpha)}\right)^{1/\alpha} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha}.$$

Also, the production function implies

$$\frac{\bar{Y}}{\bar{K}} = \left(\frac{\bar{A}\bar{N}}{\bar{K}}\right)^\alpha \approx \left(\frac{r+\delta}{1-\alpha}\right).$$

Given the budget constraint, given in (3.1), the consumption-capital ratio is also constant along the balanced growth path and equal to the value derived in section 3.2.1.

Log-Linear System

Linearizing the production function, we obtain

$$y_t = \alpha(a_t + n_t) + (1-\alpha)k_{t-1}.$$

Following the same techniques that we used to derive (3.9) and noting that output is now a function of the endogenous labor choice, we obtain

$$k_t = \lambda_1 k_{t-1} + \lambda_2(a_t + n_t) + [1 - \lambda_1 - \lambda_2]c_t, \tag{3.21}$$

where λ_1 and λ_2 are the same as before. The only difference from (3.9) is that λ_2 multiplies n_t as well as a_t. The interest rate is now

$$r_{t+1} = \lambda_3(a_{t+1} + n_{t+1} - k_t),$$

and the log-linear version of the Euler equation, given in (3.12), becomes

$$E_t[\Delta c_{t+1}] = \lambda_3 E_t[a_{t+1} + n_{t+1} - k_t]. \tag{3.22}$$

The only difference from (3.12) is that σ is now equal to 1 and n_{t+1} appears in the equation. To linearize the first-order condition for labor, first rewrite (3.20) as

$$\theta(1-N_t)^{-\gamma_N} = \alpha\left(\frac{A_t N_t}{K_{t-1}}\right)^{\alpha}\frac{K_{t-1}}{C_t}\frac{1}{N_t},$$

so that each term is constant along the balanced growth path. Linearizing this result implies

$$-\gamma_N\theta(1-\bar{N})^{-\gamma_N-1}\bar{N}n_t = \alpha\left(\frac{\bar{A}\bar{N}}{\bar{K}}\right)^{\alpha}\frac{\bar{K}}{\bar{C}}\frac{1}{\bar{N}}[\alpha(a_t+n_t-k_{t-1})+(k_{t-1}-c_t)-n_t].$$

Thus, after rearranging, we obtain

$$n_t = \left(\frac{1-\bar{N}}{\bar{N}}\right)\sigma_N\left(\alpha a_t + (1-\alpha)(k_{t-1}-n_t) - c_t\right), \tag{3.23}$$

where $\sigma_N = 1/\gamma_N$. Rewrite (3.23) as

$$n_t = \nu[(1-\alpha)k_{t-1} + \alpha a_t - c_t], \tag{3.24}$$

where

$$\nu = \nu(\sigma_N) \equiv \frac{(1-\bar{N})\sigma_N/\bar{N}}{1+(1-\alpha)(1-\bar{N})\sigma_N/\bar{N}} \equiv \frac{\eta}{1+\eta},$$

where η is the Frisch elasticity of labor supply (i.e. $(\partial N/\partial w)(w/N)|_\lambda$). Thus, the coefficient ν measures the elasticity of labor supply to shocks that change the real wage, taking account of the fact that demand for labor is downward sloping. As the curvature of the utility function for leisure increases, ν falls and becomes 0 when γ_N is infinite.

Using (3.24) to substitute n_t out of (3.21) and (3.22), we obtain

$$k_t = (\lambda_1 + \lambda_2\nu(1-\alpha))k_{t-1} + \lambda_2(1+\alpha\nu)a_t + (1-\lambda_1-\lambda_2(1+\nu))c_t \tag{3.25}$$

$$(1+\lambda_3\nu)E_t c_{t+1} = c_t + \lambda_3(1+\alpha\nu)E_t a_{t+1} - \lambda_3(1-\nu(1-\alpha))k_t. \tag{3.26}$$

Once again log consumption is linear in log capital and log technology with coefficients η_{ck} and η_{ca}. Guess that c_t and k_t are given by

$$c_t = \eta_{ck}k_{t-1} + \eta_{ca}a_t,$$
$$k_t = \eta_{kk}k_{t-1} + \eta_{ka}a_t,$$

where

$$\begin{aligned}\eta_{kk} &= \lambda_1 + \lambda_2\nu(1-\alpha) + \eta_{ck}[1-\lambda_1-\lambda_2(1+\nu)],\\ \eta_{ka} &= \lambda_2(1+\alpha\nu) + \eta_{ca}[1-\lambda_1-\lambda_2(1+\nu)].\end{aligned} \tag{3.27}$$

To solve for the coefficients, first plug in the guess for c_t and $E_t c_{t+1}$ into (3.26) to obtain

$$(1+\lambda_3\nu)(\eta_{ck}k_t + \eta_{ca}E_t a_{t+1}) = (\eta_{ck}k_{t-1} + \eta_{ca}a_t) + \lambda_3(1+\alpha\nu)E_t a_{t+1} - \lambda_3(1-\nu(1-\alpha))k_t.$$

Collecting terms, we obtain

$$[(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\alpha))]k_t + [(1+\lambda_3\nu)\eta_{ca} - \lambda_3(1+\alpha\nu)]E_t a_{t+1} = \eta_{ck}k_{t-1} + \eta_{ca}a_t. \quad (3.28)$$

Using the guess for k_t and the equation for η_{kk}, given in (3.27), after equating coefficients on k_{t-1}, we obtain

$$[(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\alpha))][\lambda_1 + \lambda_2\nu(1-\alpha) + \eta_{ck}(1-\lambda_1-\lambda_2(1+\nu))] - \eta_{ck} = 0$$

Thus,

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0,$$

where

$$\begin{aligned}
Q_2 &= [1+\lambda_3\nu][1-\lambda_1-\lambda_2(1+\nu)],\\
Q_1 &= [1+\lambda_3\nu][\lambda_1+\lambda_2\nu(1-\alpha)] + \lambda_3[1-\nu(1-\alpha)][1-\lambda_1-\lambda_2(1+\nu)] - 1,\\
Q_0 &= \lambda_3[1-(1-\alpha)\nu][\lambda_1+\lambda_2\nu(1-\alpha)].
\end{aligned}$$

The solution is given by the quadratic formula as before.

Noting that $E_t a_{t+1} = \varphi a_t$, equating coefficients on a_t in (3.28) implies

$$[(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\alpha))]\eta_{ka} + \varphi[(1+\lambda_3\nu)\eta_{ca} - \lambda_3(1+\alpha\nu)] - \eta_{ca} = 0.$$

Using the equation for η_{ka}, given in (3.27), the solution for η_{ca} is

$$\eta_{ca} = \frac{(1+\alpha\nu)[\varphi\lambda_3 - \lambda_2((1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\alpha)))]}{\varphi(1+\lambda_3\nu) - 1 + [(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\alpha))][1-\lambda_1-\lambda_2(1+\nu)]}.$$

Substituting the guess for consumption into (3.24), we obtain

$$\begin{aligned}
n_t &= \nu\left[(1-\alpha)k_{t-1} + \alpha a_t - (\eta_{ck}k_{t-1} + \eta_{ca}a_t)\right]\\
&= \nu(1-\alpha-\eta_{ck})k_{t-1} + \nu(\alpha-\eta_{ca})a_t\\
&= \eta_{nk}k_{t-1} + \eta_{na}a_t, \qquad (3.29)
\end{aligned}$$

where

$$\eta_{nk} = \nu(1-\alpha-\eta_{ck}) \qquad \eta_{na} = \nu(\alpha-\eta_{ca}).$$

Higher k_{t-1} implies w_t increases by $(1-\alpha)$, which stimulates labor supply. However, higher k_{t-1} also increases c_t, which has an offsetting effect on labor supply (higher consumption lowers the marginal utility of income and reduces work effort). Therefore the net effect is $(1-\alpha-\eta_{ck})$. Higher a_t implies w_t increases by α, but it also raises c_t. Therefore, the net effect is $(\alpha-\eta_{ca})$.

Recall the log-linear equation for y_t and substitute in (3.29)

$$\begin{aligned}
y_t &= \alpha a_t + \alpha n_t + (1-\alpha)k_{t-1}\\
&= \alpha a_t + \alpha\eta_{na}a_t + \alpha\eta_{nk}k_{t-1} + (1-\alpha)k_{t-1}\\
&= [\alpha + \alpha\eta_{na}]a_t + [1-\alpha+\alpha\eta_{nk}]k_{t-1}.
\end{aligned}$$

Plugging in the definition for η_{na} and η_{nk}, we obtain

$$\begin{aligned} y_t &= [\alpha + \alpha\nu(\alpha - \eta_{ca})]a_t + [(1-\alpha) + \alpha\nu(1-\alpha-\eta_{ck})]k_{t-1} \\ &= \eta_{ya}a_t + \eta_{yk}k_{t-1}, \end{aligned}$$

where

$$\eta_{ya} = \alpha + \alpha\nu(\alpha - \eta_{ca}) \qquad \eta_{yk} = (1-\alpha) + \alpha\nu(1-\alpha-\eta_{ck})$$

Once again, output follows an ARMA(2,1) process. Notice that variable labor amplifies the initial output response to a technology shock—instead of α, it is now $\alpha + \alpha\nu(\alpha - \eta_{ca})$.

As σ_N rises (labor becomes more substitutable over time)

- η_{nk} becomes increasingly negative, since consumption rises more than the real wage.
- η_{na} becomes increasingly positive, since consumption rises less than the real wage.

η_{ck} is independent of φ. η_{na} declines with φ, since a persistent technology shock raises consumption more than a transitory shock.

The output multipliers for technology are given by η_{ya}. With a fixed labor supply, $\eta_{ya} = \alpha = 2/3$. With a variable labor supply, $\eta_{ya} = \alpha + \alpha\nu(\alpha - \eta_{ca})$, which can exceed unity for high enough values of σ_N. However, η_{ya} falls as φ rises. Values of $\eta_{ya} > 1$ are significant because they allow absolute declines in output to be generated by positive but slower than normal technology growth. Therefore, we do not need absolute declines in technology to get below trend growth in output.

Chapter 4

Money and Policy

4.1 Fiat Currency in a Lucas Tree Model

This section is based on Sargent (1987). Fiat currency is intrinsically useless—it does not enter utility or production. Moreover, it is not backed by any commodity that has intrinsic value and it earns zero interest, which means that it is dominated in rate of return by all other assets. Hence, in a standard model without any frictions, money is not valued.

To see this more clearly, first consider the following variant of the Lucas (1978) tree model. There are a large number of identical households each endowed with a single, identical, non-depreciating fruit tree, $s_0 = 1$. A fruit tree produces dividends (fruit), y_t, according to some exogenous stochastic process. Dividends cannot be stored—the only store of value are trees. We are interested in pricing the assets (trees). Denote the price of a tree by p_t, measured in units of consumption goods per tree. Each household chooses sequences $\{c_t, s_{t+1}\}_{t=0}^{\infty}$ to maximize expected lifetime utility, given by,

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1, \tag{4.1}$$

subject to

$$c_t + p_t s_{t+1} = (p_t + y_t) s_t.$$

The first order conditions imply

$$p_t = \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + y_{t+1}) \right\},$$

which we can iterate forward to obtain

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} y_{t+j}.$$

Money (fiat currency) is an asset that pays no dividends, so asset pricing reasoning would imply money has no value. Now consider a version of Lucas's tree model in which the government places M units of fiat currency into circulation. The government's budget constraint is

$$\begin{aligned} g_0 &= M w_0 \\ g_t &= 0, \quad t \geq 1, \end{aligned}$$

where g_0 are government purchases at $t = 0$ and w_t is the value of currency, measured in goods at time t per unit of currency. Once again, assume households are endowed with $s_0 = 1$ fruit tree. The consumer maximizes (4.1) subject to

$$c_t + p_t s_{t+1} + w_t m_{t+1} \leq s_t(p_t + y_t) + w_t m_t,$$

where m_t is the amount of currency owned at the beginning of period t. The first order conditions are given by

$$\begin{gathered} u'(c_t) = \lambda_t \\ \lambda_t p_t = \beta\lambda_{t+1}(p_{t+1} + y_{t+1}) \\ -\lambda_t w_t + \beta\lambda_{t+1} w_{t+1} \leq 0, \quad m_{t+1} \geq 0, \quad \text{(with “CS”)}, \end{gathered}$$

where λ_t is the Lagrange multiplier on the time t budget constraint. Consolidating, we obtain

$$\begin{gathered} \beta\frac{u'(c_{t+1})}{u'(c_t)}\left(\frac{p_{t+1} + y_{t+1}}{p_t}\right) = 1 \\ m_{t+1}\left[w_t - \beta\frac{u'(c_{t+1})}{u'(c_t)} w_{t+1}\right] = 0. \end{gathered}$$

The market clearing conditions are given by

$$\begin{aligned} m_{t+1} &= M, \quad t \geq 0 && \text{(money demand = money supply)} \\ c_0 + g_0 &= c_0 + Mw_0 = y_0, && (t = 0, \text{resource constaint}) \\ c_t &= y_t. && (t \geq 1, \text{resource constaint}) \end{aligned}$$

We want to show that given a bounded dividend stream, $w_0 = 0$ is the only price of money consistent with equilibrium. If $w_i = 0$ for some i, then $w_t = 0$ for all t. Thus, for $w_0 > 0$, we need $w_t > 0$ for all t. We will suppose this is true and derive a contradiction. The intuition is that if w is zero tomorrow, then today I will not demand any money since I will have no reason to hold money into the next period. Also, since the value of money tomorrow is dependent on the value of money today, we know that if money has no value in a particular period it will not have any value in all subsequent periods.

Define

$$R_t = \frac{p_{t+1} + y_{t+1}}{p_t} \qquad \rightarrow \qquad \beta\frac{u'(c_{t+1})}{u'(c_t)} R_t = 1.$$

Since $m_{t+1} = M > 0$ in competitive equilibrium

$$w_t = \beta\frac{u'(c_{t+1})}{u'(c_t)} w_{t+1} \qquad \rightarrow \qquad w_{t+1} = R_t w_t, \quad t \geq 0.$$

Iterating implies

$$w_t = \left(\prod_{j=0}^{t-1} R_j\right) w_0 = \left(\prod_{j=0}^{t-1} \frac{u'(c_j)}{\beta u'(c_{j+1})}\right) w_0 = \frac{u'(c_0)}{\beta^t u'(c_t)} w_0.$$

After imposing market clearing, we obtain

$$w_t = \beta^{-t}\frac{u'(y_0 - Mw_0)}{u'(y_t)} w_0.$$

If $w_0 > 0$, then $\{w_t\}_{t=0}^{\infty}$ grows without bound. If w_t were to grow without bound, the budget constraint implies that the consumer's initial wealth at t, $(p_t + y_t) + w_t M$, would grow without bound. If this were so $c_t = y_t$ for $t \geq 1$ would not be an optimal consumption plan. Therefore, such a path for w_t cannot be an equilibrium, and we cannot have $w_0 > 0$. We can conclude that fiat currency is valueless in this economy ($w_t = 0\ \forall t$), as asset pricing theory would suggest.

4.2 Fiscal and Monetary Theories of Inflation

This section is based on Ljungqvist and Sargent (2012). Consider a model without uncertainty, but where households must shop to acquire consumption goods. There is a constant endowment, y, each period, which can be divided between private consumption, c_t, and government purchases, g_t. The aggregate resource constraint is given by

$$c_t + g_t = y.$$

The household chooses sequences to maximize lifetime utility, given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t), \tag{4.2}$$

where ℓ_t is leisure. Given one unit of time each period

$$\ell_t + s_t = 1, \tag{4.3}$$

where s_t is the amount of time spent shopping, which is required to purchase a particular level of consumption. The shopping or transactions technology is

$$s_t = H\left(c_t, \frac{m_t}{p_t}\right), \tag{4.4}$$

where $H, H_c, H_{cc}, H_{m/p,m/p} \geq 0$ and $H_{m/p}, H_{c,m/p}, \leq 0$. These restrictions imply that 1. a household cannot spend a negative amount of time shopping; 2. higher consumption requires more time shopping, at an increasing rate; 3. an increase in real money balances reduces shopping time but at a decreasing rate, and 4. the shopping costs of higher consumption are decreasing in real money balances. An example of this technology is

$$H\left(c_t, \frac{m_t}{p_t}\right) = \frac{c_t}{m_t/p_t}\varepsilon, \qquad \varepsilon > 0. \tag{4.5}$$

Think of ε as the fixed time cost of driving to the bank and its coefficient as the number of trips to the bank.

The budget constraint is

$$c_t + \frac{b_t}{R_t} + \frac{m_t}{p_t} = y - \tau_t + b_{t-1} + \frac{m_{t-1}}{p_t}, \tag{4.6}$$

where m_t are dollars held from t to $t+1$, p_t is the price level, expressed in dollars per consumption goods, τ_t is a lump-sum tax, and b_t is a one-period risk-free real bond with gross return R_t. Updating (4.6) one period and solving for b_t, we obtain

$$b_t = c_{t+1} + \frac{b_{t+1}}{R_{t+1}} + \frac{m_{t+1}}{p_{t+1}} - y + \tau_{t+1} - \frac{m_t}{p_{t+1}}. \tag{4.7}$$

Substituting (4.7) into (4.6), we obtain

$$c_t + \frac{c_{t+1}}{R_t} + \left[1 - \frac{R_{mt}}{R_t}\right]\frac{m_t}{p_t} + \frac{b_{t+1}}{R_t R_{t+1}} + \frac{m_{t+1}/p_{t+1}}{R_t} = y - \tau_t + \frac{y - \tau_{t+1}}{R_t} + b_{t-1} + \frac{m_{t-1}}{R_t},$$

where $R_{mt} = p_t/p_{t+1}$. If $1 - R_{mt}/R_t < 0$, then the agent could achieve infinite consumption by choosing $m_t/p_t = \infty$, which obviously can not be the case. Thus, to ensure a bounded budget set,

$$1 - \frac{R_{mt}}{R_t} = \frac{i_t}{1 + i_t} \geq 0,$$

where $1 + i_t = R_t/R_{mt}$ is the gross nominal interest rate. If the real return on money, R_{mt}, was larger than the real return on bonds, R_t, then households could just take out loans to hold more money and earn arbitrarily large profits. This is equivalent to saying that in this case households are worse off investing in bonds over holding money, which is the same as saying the net nominal interest rate is negative.

The household maximizes (4.2), subject to (4.3), (4.4), and (4.6). The first order conditions are given by

$$c_t: \quad U_c(t) - \lambda_t - \mu_t H_c(t) = 0 \tag{4.8}$$

$$\ell_t: \quad U_\ell(t) - \mu_t = 0 \tag{4.9}$$

$$b_t: \quad -\frac{\lambda_t}{R_t} + \beta\lambda_{t+1} = 0 \tag{4.10}$$

$$m_t: \quad -\frac{\lambda_t}{p_t} - \frac{\mu_t}{p_t}H_{m/p}(t) + \beta\frac{\lambda_{t+1}}{p_{t+1}} = 0. \tag{4.11}$$

Using (4.8) and (4.9), we obtain

$$\underbrace{\lambda_t}_{\substack{\text{Shadow Price}\\ \text{of Wealth}}} = \underbrace{u_c(t)}_{\substack{\text{Marginal Utility}\\ \text{of Consumption}}} - \underbrace{u_\ell(t)H_c(t)}_{\substack{\text{Marginal Disutility of having}\\ \text{to shop for Consumption}}}. \tag{4.12}$$

Plugging this result into (4.10) yields

$$R_t = \frac{1}{\beta}\left[\frac{u_c(t) - u_\ell(t)H_c(t)}{u_c(t+1) - u_\ell(t+1)H_c(t+1)}\right]. \tag{4.13}$$

Combining (4.10) with (4.11) implies

$$\underbrace{\frac{R_t - R_{mt}}{R_t}\lambda_t}_{\substack{\text{Marginal cost of an}\\ \text{additional unit of money}}} = \underbrace{-\mu_t H_{m/p}(t)}_{\substack{\text{Marginal benefit of an}\\ \text{additional unit of money}}}. \tag{4.14}$$

The cost of holding money balances instead of bonds is the lost interest earnings $R_t - R_{mt}$ discounted at rate R_t and expressed in time t utility when multiplied by the shadow price λ_t. The benefit of an additional unit of real money balances is the savings in shopping time $-H_{m/p}(t)$ evaluated at the shadow price μ_t. After plugging in for λ_t and μ_t using (4.9) and (4.12), we obtain

$$\left(1 - \frac{R_{mt}}{R_t}\right)\left[\frac{u_c(t)}{u_\ell(t)} - H_c(t)\right] + H_{m/p}(t) = 0. \tag{4.15}$$

Since $\ell_t = 1 - H(c_t, m_t/p_t)$, (4.15) implicitly defines the real demand for money, given by,

$$\frac{m_t}{p_t} = F\left(c_t, \frac{R_{mt}}{R_t}\right) = \tilde{F}(c_t, i_t), \tag{4.16}$$

where $F_c > 0$, $F_{Rm/R} > 0$ and $\tilde{F}_i < 0$, which can be verified by applying the implicit function theorem to (4.15).

The government finances exogenous sequences $\{\tau_t, g_t\}_{t=0}^{\infty}$ through seigniorage revenues and one-period government debt. The government's flow budget constraint is given by

$$g_t = \tau_t + \frac{B_t}{R_t} - B_{t-1} + \frac{M_t - M_{t-1}}{p_t}, \tag{4.17}$$

where B_t and M_t are the supplies of real government bonds and currency issued to the private sector (M_{-1}, B_{-1} given) and g_t is government spending. In equilibrium $b_t = B_t$, $m_t = M_t$, the government's budget constraint holds, and $c_t + g_t = y$.

4.2.1 Policy Experiments

We need a complete specification of government policy. We will study government policies that distinguish between the "short run" (initial date) and the "long run" (stationary equilibrium). Assume

$$\begin{aligned} g_t &= g, && \forall t \geq 0, \\ \tau_t &= \tau, && \forall t \geq 1, \\ B_t &= B, && \forall t \geq 0. \end{aligned}$$

We permit $\tau_0 \neq \tau$ and $B_{-1} \neq B$. This means the economy is in a stationary equilibrium for $t \geq 1$, but starts in a different position for $t = 0$. This approach reduces the dynamics to 2 periods: now ($t = 0$) and the future ($t \geq 1$).

4.2.2 Stationary Equilibrium

We seek an equilibrium with

$$\begin{aligned} \frac{p_t}{p_{t+1}} &= R_m, && t \geq 0, \\ R_t &= R, && t \geq 0, \\ c_t &= c, && t \geq 0, \\ s_t &= s, && t \geq 0. \end{aligned} \tag{4.18}$$

Substituting (4.18) into (4.13) and (4.16), we obtain

$$\begin{aligned} R &= \frac{1}{\beta}, \\ \frac{M_t}{p_t} &= F\left(c, \frac{R_m}{R}\right) \equiv f(R_m), \end{aligned} \tag{4.19}$$

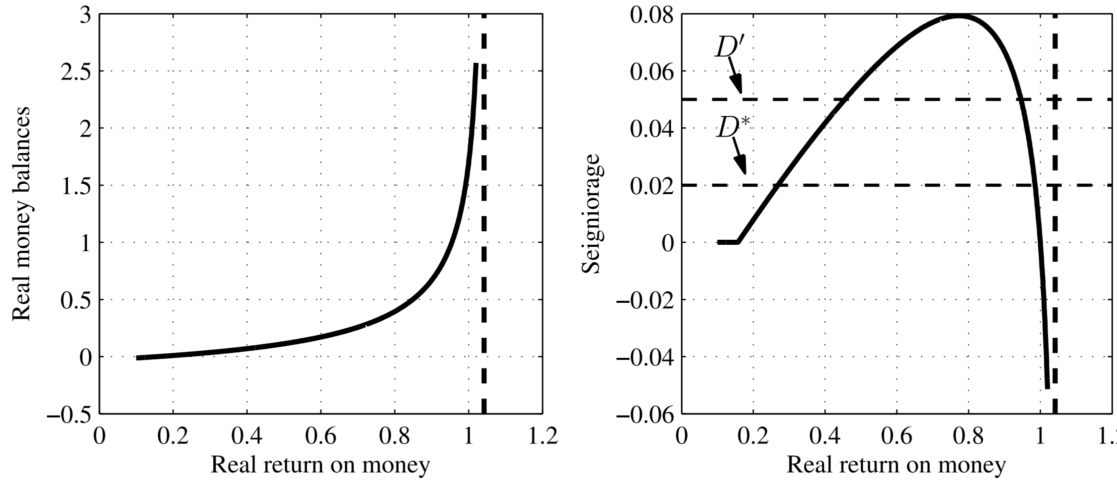

Figure 4.1: Price determination

where $f'(R_m) > 0$. There are two equilibrium conditions. First impose equilibrium on the government budget constraint at $t \geq 1$ (i.e., the future) to obtain

$$\begin{aligned} g - \tau + \frac{B(R-1)}{R} &= \frac{M_t}{p_t} - \frac{M_{t-1}}{p_t} \\ &= f(R_m) - \frac{M_{t-1}}{p_t}\frac{p_{t-1}}{p_{t-1}} \\ &= f(R_m) - f(R_m)R_m \\ &= \underbrace{f(R_m)(1-R_m)}_{\text{seigniorage}}, \end{aligned} \tag{4.20}$$

where $g - \tau$ represents the net of interest (primary) deficit and $g - \tau + B(R-1)/R$ represents the gross deficit. Then impose equilibrium on the government budget constraint at $t = 0$ (i.e., the current period) to obtain

$$\frac{M_{-1}}{p_0} = f(R_m) - (g + B_{-1} - \tau_0) + \frac{B}{R}. \tag{4.21}$$

Given (g, τ, B), (4.20) pins down R_m (i.e., the inflation rate). Given (g, τ_0, B) and initial conditions (B_{-1}, M_{-1}), (4.21) pins down the initial price level, p_0. Thus, (4.20) and (4.21) recursively determine the equilibrium price sequence, $\{p_t\}_{t=0}^{\infty}$.

It is useful to illustrate the determination of an equilibrium with a parametric example. Let the utility function and the transactions technology be given by

$$\begin{aligned} u(c,\ell) &= \frac{c^{1-\delta}}{1-\delta} + \frac{\ell^{1-\alpha}}{1-\alpha}, \\ H\left(c, \frac{m}{p}\right) &= \frac{c}{1+m/p}, \end{aligned}$$

where the latter is a modified version of (4.5), so that transactions can be carried out even in the absence of money. For the parameter values $(\beta, \delta, \alpha, c) = (0.96, 0.7, 0.5, 0.4)$, Figure 4.1 plots real money balances and seigniorage revenues as a function of the real return on money, R_m. Notice that as the real return on money approaches $R = 1/\beta$, real money demand rises sharply. Real money demand cannot exceed $1/\beta$, since $R_m < R$. In steady state, $m/p = f(R_m)$. Thus, given real money demand, we can solve for seigniorage revenues, which equal $f(R_m)(1 - R_m)$. As the real return on money rises, seigniorage revenues initially rise, but sharply decline as R_m approaches $1/\beta$. This is because the government is continuously withdrawing money from circulation to raise the real return on money above 1.

4.2.3 Monetary Doctrines

Now consider alternative policies and how they affect price-level determination.

1. *Quantity Theory of Money*. Suppose money is injected via a "helicopter drop": the initial nominal money stock changes from M_{-1} to λM_{-1}, $\lambda > 1$ holding (τ_0, τ, g, B) fixed. With fiscal policy unchanged, the LHS of (4.20) is unchanged. Hence, R_m (and inflation) is unchanged. Thus, the RHS of (4.21) is unchanged and p_0 must rise to λp_0 so that $\frac{M_{-1}}{p_0}$ is unchanged. The entire sequence of prices $\{p_t\}_{t=0}^{\infty}$ since R_m is unchanged. The "helicopter drop" is neutral.

2. *Sustained Deficits Cause Inflation*. Suppose

$$D^* = g - \tau + \frac{B(R-1)}{R}, \qquad D' > D^*.$$

   It is clear from figure 4.1 that D^* can be financed with either a low or high rate of return on money (i.e., high inflation or low inflation). Recall that seigniorage $s = f(R_m)(1 - R_m)$, which equals the monetary base, $f(R_m)$, times the rate of nominal money growth, $1 - R_m$ (i.e., the inflation tax rate). The "normal" side of the Laffer curve is where an increase in the tax rate, $1 - R_m$, leads to an increase in seigniorage revenues. Thus, the higher R_m that solves (4.20) is on the "normal" of the Laffer curve ($R_m \uparrow \rightarrow (1 - R_m) \downarrow \rightarrow s \downarrow$). On this side of the Laffer curve, a higher deficit ($D' > D^*$) implies a lower real rate of return on money ($R'_m < R^*_m$) and higher inflation. We will always select the higher R_m that solves (4.20), since the lower rate of return has "perverse" dynamics in the sense that higher deficits reduce inflation.

3. *Zero Inflation Policy*. Zero inflation ($\pi = 1$) implies $R_m = 1$. Thus, (4.20) implies

$$g - \tau + \frac{B(R-1)}{R} = 0$$

   or

$$B = \frac{R}{R-1}(\tau - g) = \underbrace{\sum_{t=0}^{\infty} R^{-t}(\tau - g)}_{\substack{\text{Present Value of} \\ \text{all Future Surpluses}}}.$$

   The equation says that the real value of debt equals the present value of the net of interest government surpluses. Without the aid of seigniorage, the zero inflation policy implies a restriction on fiscal policy.

4. *Unpleasant Monetarist Arithmetic*. Assume the conventional side of Laffer curve and consider an open market sale of bonds at time 0, holding fiscal policy constant. That is,

$$-\Delta M_0 = \Delta B_0 > 0,$$

with (g, τ_0, τ) fixed. At $t = 0$, $\Delta B_0 > 0$ in (4.20) and (4.21). Higher debt implies high debt service (i.e., by $(R-1)\Delta B_0/R$) in the future. Hence, (4.20) implies

$$f(R_m)(1-R_m) \uparrow \rightarrow R_m \downarrow \rightarrow \pi \uparrow .$$

However, the effect on p_0 can be anything—it depends on $f'(R_m)$. Since $f'(R_m) > 0$, a decrease in R_m will decrease the RHS of (4.21), but an increase in B_0 will increase the RHS of (4.21). Thus, the change in p_0 depends on magnitude of the changes to the RHS of (4.21). If the RHS change is positive, meaning the change in bonds dominates, then p_0 must fall (usual result). If the RHS change is negative, meaning the change in R_m dominates, then p_0 must rise. Tighter money via open market operations—at best—temporarily lowers p but at the cost of permanently higher p. Sargent and Wallace (1981) call this "unpleasant monetarist arithmetic".

5. *An Open Market Operation that is Neutral*. Redefine an open market operation (OMO) from the definition used in unpleasant monetarist arithmetic. Give the monetary authority (MA) fiscal powers so that OMOs have effects like those in the quantity theory experiment. Consider an OMO that decreases M_0 and increases B and τ such that

$$\frac{\Delta B(R-1)}{R} = \Delta\tau, \tag{4.22}$$

and $\Delta\tau_0 = 0$. If future taxes satisfy (4.22) for $t \geq 1$, then (4.20) is satisfied at the initial R_m (like the zero inflation policy). Equation (4.22) implies that lump sum taxes in the future adjust by exactly enough to service additional interest payments arising from the OMO's effects on B:

$$\underbrace{M_0 \downarrow,\ B \uparrow}_{\text{OMO}} \rightarrow \tau \uparrow .$$

In short, this policy causes proportionate decreases in the money supply and the price level, leaves R_m unaltered, and fulfills the quantity theory of money.

6. *The Optimum Quantity of Money*. Given stationary (g, B), Friedman (1969) argues that agents are better off with higher real balances (higher R_m). With a sufficiently large gross of interest surplus (i.e., $g - \tau + B(R-1)/R < 0$), the government can attain $R_m \in (1, 1/\beta)$. Given (g, B), a τ can be chosen to obtain the required surplus to hit the target R_m. The proceeds of the tax are used to retire currency from circulation, which generates deflation and makes the real return on money equal to the target value of R_m. Thus, this policy replaces the inflation tax with non-distorting lump-sum taxes, thus pursuing Friedman's optimal policy of saturating the economy with real balances.

   The social value of real balances in the model comes from reducing shopping time. The optimal quantity of M minimizes the time spent shopping. Suppose there is a satiation point in real balances, $\psi(c)$, for any c such that

$$H_{\frac{m}{p}}\left(c, \frac{m_t}{p_t}\right) = 0 \quad \text{for} \quad \frac{m_t}{p_t} \geq \psi(c).$$

According to (4.14), the government can achieve this optimal allocation only by choosing $R_m = R$, since $\lambda_t, \mu_t > 0$. Thus, welfare is at its maximum when the economy is satiated with real balances (shopping time is minimized).

7. *A Ricardian Experiment*. Consider a debt financed tax cut, where future taxes adjust to service the debt. Monetary policy is held constant, so there is no change in the sequence of nominal balances $\{M_t\}_{t=0}^{\infty}$. That is,

$$\Delta\tau_0 = \Delta B/R, \qquad \Delta\tau = (R-1)\Delta B/R,$$

and (4.20) and (4.21) are satisfied at the initial R_m and p_0. With this policy, the initial trax cut is neutral—lump-sum taxes in the future adjust by just enough to service any additional interest payments arising from the tax cuts effects on B. Of, course, the assumption of lump-sum taxes is essential. If the tax cut was financed by taxes levied proportionally on income, the tax cut would not be neutral.

8. *Fiscal Theory of the Price Level (FTPL)* (Ljungqvist and Sargent version). In previous doctrines about inflation, the government sets (g, τ_0, τ, B) and (B_{-1}, M_{-1}) are inherited from the past, which implies that the model determines (R_m, p_0). More specifically, given (g, B, τ), (4.20) implies R_m. Then given (g, τ_0, B, R_m), (4.21) implies p_0. With p_0, M_0 is determined by $f(R_m) = M_0/p_0$. In this setting, the government commits to $g - \tau + B(R-1)/R$ and the market determines p_0, R_m.

   The FTPL changes the assumptions about which variables the government sets. In this case, the government sets the present value of seigniorage, $f(R_m)(1-R_m)/(R-1)$, so that B is endogenous and the government commits to peg the rate of inflation R_m^{-1} (and the nominal interest rate). To illustrate this argument, rewrite (4.20) as

$$\begin{aligned}\frac{B}{R} &= \frac{1}{R-1}\left[(\tau - g) + f(R_m)(1-R_m)\right] \\ &= \sum_{t=1}^{\infty} R^{-t}(\tau - g) + f(R_m)\left(\frac{1-R_m}{R-1}\right). \end{aligned} \tag{4.23}$$

   Then substitute (4.23) into (4.21) to obtain

$$\begin{aligned}\frac{M_{-1}}{p_0} + B_{-1} &= \sum_{t=0}^{\infty} R^{-t}(\tau - g) + f(R_m)\left(1 + \frac{1-R_m}{R-1}\right) \\ &= \sum_{t=0}^{\infty} R^{-t}(\tau - g) + \sum_{t=1}^{\infty} R^{-t} f(R_m)(R - R_m). \end{aligned} \tag{4.24}$$

   In stationary equilibrium: $R = 1/\beta$. Thus, $(1+i)\beta = R_m^{-1} = \pi$. This means that pegging the nominal interest rate is equivalent to pegging the inflation rate and the present value of seigniorage. The government chooses (g, τ, τ_0, R_m). Then (4.23) determines B as the present value of future surpluses, including seigniorage revenues, and (4.24) determines p_0. Finally, the endogenous quantity of money is determined by

$$\frac{M_0}{p_0} = f(R_m).$$

   In the unpleasant monetarist arithmetic doctrine, the focus is on how the inflation tax responds to fiscal conditions that the government inherits. In the FTPL, the inflation tax is fixed by pegging the nominal interest rate. This forces other aspects of fiscal policy and the price level to adjust.

4.2.4 Summary

These doctrines, those simple, highlight the centrality of monetary and fiscal policy interactions for the nature of equilibrium. Although this general point is well-known, the profession sometimes ignores it, since prescribing monetary and fiscal policy is much more difficult than prescribing monetary policy and assuming fiscal policy will adjust to ensure fiscal sustainability.

4.3 Monetary and Fiscal Policy Interactions

This section is based on Leeper (1991) and Leeper and Yun (2006). Consider an endowment economy with lump-sum taxes. Each period an infinitely-lived representative household is endowed with a constant quantity y of consumption goods and chooses quantities $\{c_t, M_t, B_t\}_{t=0}^{\infty}$ to maximize expected lifetime utility, given by,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(M_t/P_t)\right]$$

subject to

$$c_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} + \tau_t \leq y + \frac{M_{t-1}}{P_t} + \frac{R_{t-1}B_{t-1}}{P_t}, \tag{4.25}$$

where M_t is nominal money balances, B_t is a one-period nominal bond that pays R_t dollars at $t+1$, and τ_t is a lump-sum tax.

The government chooses $\{M_t, B_t, \tau_t\}$ to finance a constant level of purchases of goods, g, to satisfy the government's budget constraint

$$g = \tau_t + \frac{M_t - M_{t-1}}{P_t} + \frac{B_t - R_{t-1}B_{t-1}}{P_t}. \tag{4.26}$$

The first order necessary conditions are given by

$$u'(c_t) = \lambda_t, \tag{4.27}$$

$$v'\left(\frac{M_t}{P_t}\right)\frac{1}{P_t} - \frac{\lambda_t}{P_t} + \beta E_t\left\{\frac{\lambda_{t+1}}{P_{t+1}}\right\} = 0, \tag{4.28}$$

$$-\frac{\lambda_t}{P_t} + \beta R_t E_t\left\{\frac{\lambda_{t+1}}{P_{t+1}}\right\} = 0, \tag{4.29}$$

where λ_t is the Lagrange multiplier on (4.25). After imposing market clearing ($c_t = c = y - g$), the optimality conditions imply equilibrium Fisher and money demand relations, given by,

$$\frac{1}{R_t} = \beta E_t\left\{\frac{1}{\pi_{t+1}}\right\}, \tag{4.30}$$

$$\frac{v'(m_t)}{u'(c_t)} = \frac{R_t - 1}{R_t}, \tag{4.31}$$

where $\pi_t = P_t/P_{t-1}$ is the gross inflation rate and $m_t = M_t/P_t$ is real money balances.

Government policies follow simple rules. The monetary authority sets the nominal interest rate as a function of inflation,

$$R_t = e^{\alpha_0}\pi_t^{\alpha}\theta_t, \tag{4.32}$$

and the fiscal authority sets taxes as a function of debt,

$$\tau_t = e^{\gamma_0} b_{t-1}^{\gamma} \psi_t, \tag{4.33}$$

where $\theta_t \in [\underline{\theta}, \overline{\theta}]$ and $\psi_t \in [\underline{\psi}, \overline{\psi}]$ are shocks with unit mean. How can we interpret θ_t and ψ_t? They are exogenous changes in policy unrelated to economic conditions. Think of θ_t as an open market operation (i.e $\theta_t > 0$ is an open market sale, $B \uparrow, M \downarrow$) and ψ_t as a debt financed tax shock (i.e., $\psi_t > 0$ implies a tax increase, $\tau \uparrow, B \downarrow$).

Equations (4.25), (4.26), and (4.30)-(4.33) reduce to a recursive system in inflation and real debt. We will work with a log-linear approximation of the equilibrium around the deterministic steady state. First, combine the Fisher equation, (4.30), with the interest rate rule, (4.32), and log-linearize to obtain

$$E_t\hat{\pi}_{t+1} = \alpha\hat{\pi}_t + \hat{\theta}_t. \tag{4.34}$$

Log-linearizing (4.31), we obtain

$$\hat{m}_t = \frac{v'(m)}{v''(m)}\frac{1}{m}\frac{1}{R-1}\hat{R}_t = \varphi\hat{R}_t = \varphi(\alpha\hat{\pi}_t + \hat{\theta}_t), \tag{4.35}$$

where φ is the interest elasticity of money demand. To see this, note that

$$\frac{dm}{dR} = \frac{u'(c)}{v''(m)}\frac{1}{R^2} = \frac{v'(m)}{v''(m)}\frac{1}{R}\frac{1}{R-1} \quad \rightarrow \quad \frac{dm}{dR}\frac{R}{m} = \frac{v'(m)}{v''(m)}\frac{1}{m}\frac{1}{R-1}.$$

The log-linear policy rules are

$$\hat{R}_t = \alpha\hat{\pi}_t + \hat{\theta}_t, \tag{4.36}$$

$$\hat{\tau}_t = \gamma\hat{b}_{t-1} + \hat{\psi}_t. \tag{4.37}$$

Focus on a steady state with no government spending ($g = 0$) and a constant price level ($\pi = 1$). In steady state, the government budget constraint, (4.26), implies $\tau/b = \beta^{-1} - 1$. After log-linearizing (4.26), we obtain

$$\tau\hat{\tau}_t + m\hat{m}_t - m\hat{m}_{t-1} + m\hat{\pi}_t - \beta^{-1}b\hat{R}_{t-1} - \beta^{-1}b\hat{b}_{t-1} + \beta^{-1}b\hat{\pi}_t + bb_t = 0.$$

Plugging in (4.35)-(4.37) and imposing steady state, yields the law of motion for real debt, given by,

$$\hat{b}_t + \lambda_1\hat{\pi}_t - [\beta^{-1} - \gamma(\beta^{-1} - 1)]\hat{b}_{t-1} + \lambda_2\hat{\pi}_{t-1} + (\beta^{-1} - 1)\hat{\psi}_t + \lambda_3\hat{\theta}_t + \lambda_4\hat{\theta}_{t-1} = 0, \tag{4.38}$$

where

$$\lambda_1 = \frac{m}{b}(\alpha\varphi + 1) + \beta^{-1}, \qquad \lambda_3 = \frac{m}{b}\varphi,$$

$$\lambda_2 = -\alpha(\frac{m}{b}\varphi + \beta^{-1}), \qquad \lambda_4 = -(\frac{m}{b}\varphi + \beta^{-1}).$$

The parameter space can be divided into four disjoint regions (see table 4.1). The eigenvalues of (4.34) and (4.38) are α and $\beta^{-1} - \gamma(\beta^{-1} - 1)$, and a unique saddle path equilibrium requires the magnitude two roots to lie on either side of 1 [Blanchard and Kahn (1980)]. Thus, Region I and Region II are the only two regions of the parameters space where unique equilibria exist.

Policy behavior in the two regions is "active" or "passive", referring to the constraints a policy authority faces.

Region	MP	FP	Name	Equilibrium
I	$\alpha > 1$	$\gamma > 1$	Active MP, Passive FP	Unique/Bounded
II	$\alpha < 1$	$\gamma < 1$	Passive MP, Active FP	Unique/Bounded
III	$\alpha < 1$	$\gamma > 1$	Passive MP, Passive FP	Indeterminacy/Sunspots
IV	$\alpha > 1$	$\gamma < 1$	Active MP, Active FP	None

Table 4.1: Regions of the parameter space

- An *active* policy authority pursues its objectives without regard to the state of government debt or the behavior of the other policy authority. Monetary policy is active in Region I and fiscal policy is active in Region II.
- A *passive* policy authority takes as given the active authority's and private sector's behavior and chooses policy to be consistent with equilibrium. Fiscal policy is passive in Region I and monetary policy is passive in Region II.

For completeness, there are two additional regions to consider. Region III, in which both policies are passive and the eigenvalues are less than one, and Region IV, in which both policies are active and the eigenvalues are greater than one.

4.3.1 Region I: Active Monetary and Passive Fiscal Policies

In Region I, monetary policy is unconstrained and actively pursues price stability by reacting strongly to inflation ($\alpha > 1$). Fiscal policy obeys the constraints imposed by private and monetary policy behavior and passively adjusts taxes to balance the budget ($\gamma > 1$). The solution for inflation comes from solving (4.34) forward

$$\hat{\pi}_t = -\frac{1}{\alpha} E_t \sum_{i=0}^{\infty} \left(\frac{1}{\alpha}\right)^i \hat{\theta}_{t+i} = -\frac{1}{\alpha}\hat{\theta}_t.$$

and the sequence for equilibrium real debt, $\{\hat{b}_t\}_{t=0}^{\infty}$, evolves according to the stable difference equation in (4.38). Inflation is entirely a monetary phenomenon in the sense that it is independent of shocks to tax policy. Fluctuations in inflation only depend on the monetary policy parameter, α, and the monetary policy shock, $\hat{\theta}_t$. However, debt and taxes respond to both monetary and tax disturbances. This region delivers Ricardian equivalence. To see this, consider a debt financed tax cut at t ($\psi_t < 0$, $\tau_t \downarrow$, $B_t \uparrow$). Since the monetary authority pins down inflation, the real level of debt (B_t/P_t) rises. With $\gamma > 1$, future taxes adjust by just enough to stabilize debt. That is, the change in the present value of future taxes equals the change in the real value of debt. When agents are infinitely lived, they bear the entire burden of the higher future taxes and fully discount any short-run benefits from the tax cut. Thus, the timing of taxes and debt is irrelevant for equilibrium allocations and prices.

4.3.2 Region II: Passive Monetary and Active Fiscal Policies

In Region II, the fiscal authority refuses to adjust taxes taxes strongly, preventing deficit shocks from being financed entirely with future taxes. Now the monetary authority obeys the constraints imposed by private and fiscal policy behavior and allows the money stock to respond to deficit shocks. Consider the special case of a pegged nominal interest rate ($\alpha = 0$) and exogenous taxes ($\gamma = 0$).

To solve for equilibrium inflation, use (4.38) to solve for lagged debt. Then apply expectations conditional on information at $t-1$, iterate forward, and plug back into (4.38) to obtain

$$\hat{\pi}_t = -\left(\frac{1-\beta}{\beta(m/b)+1}\right)\hat{\psi}_t - \frac{m/b}{m/b+\beta^{-1}}[\beta(1-\varphi)+\varphi]\hat{\theta}_t + \hat{\theta}_{t-1}.$$

A tax cut ($\hat{\psi}_t < 0$) raises inflation. Consider an open market sale of bonds, which raises the nominal interest rate ($\hat{\theta}_t > 0$). Expected inflation unambiguously rises, but current inflation may rise or fall, depending on the interest elasticity of money demand, φ. A higher level of debt relative to real money balances increases (decreases) the elasticity of $\hat{\pi}_t$ with respect to $\hat{\psi}_t$ ($\hat{\theta}_t$).

In equilibrium, real debt is given by

$$\hat{b}_t = \beta(m/b)(1-\varphi)\hat{\theta}_t.$$

Tax changes ($\hat{\psi}_t \lesseqgtr 0$) leave debt unchanged, while open market sales ($\hat{\theta}_t > 0$) raise the level of real debt. Why do debt financed tax cuts ($\psi_t < 0$, $\tau_t \downarrow$, $B_t \uparrow$) not affect the level of real debt? Without any adjustment in future taxes ($\gamma = 0$) or seigniorage ($\alpha = 0$), a tax cut makes agents feel wealthier at initial prices. Hence, the demand for goods increases and the price level rises. Prices rise until real debt (B_t/P_t) returns to its initial level. In short, B_t/P_t cannot change in response to a tax shock, because when $\alpha = \gamma = 0$ and shocks are serially uncorrelated, changes in taxes do not elicit any changes in future surpluses or seigniorage.

Finally, note that $\hat{m}_t = \varphi\hat{R}_t = \varphi\hat{\theta}_t$. Thus, tax changes also do not affect real money balances. Since P_t rises, the nominal money supply, M_t, rises passively to ensure that real money balances do not change. Without the appropriate (passive) adjustment in the money stock, there can be no equilibrium—government debt would follow an unsustainable explosive path a become worthless.

4.3.3 Region III: Passive Monetary and Passive Fiscal Policies

In region III, each policy authority acts passively, as though it is constrained to balance the budget. Suppose $\alpha < 1$. Expected inflation is pinned down by (4.34), but actual inflation is not since there is no other equation that can be used to pin down inflation. Thus, there is an entire family of solutions to (4.34) all of which are bounded, given by,

$$\hat{\pi}_t = \alpha\hat{\pi}_{t-1} + \hat{\theta}_{t-1} + \xi_t, \tag{4.39}$$

where ξ_t is a martingale difference sequence ($\xi_t = \pi_t - E_{t-1}\pi_t$), with $E_t\xi_{t+1} = 0$. Such shocks are often referred to in the literature as sunspot shocks. Any process $\{\pi_t\}$ satisfying (4.39) is consistent with equilibrium. Thus, the price level, and hence inflation and the nominal interest rate, are not determined uniquely when the interest rate rule implies a weak response of the nominal rate to changes in inflation.

4.4 Classical Monetary Model

This section is based on Galí (2008), chapter 2. Consider a simple model of a classical monetary economy. The representative household chooses sequences $\{C_t, N_t, B_t\}_{t=0}^{\infty}$ to maximize expected lifetime utility, given by,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\eta}}{1+\eta} \right\} \tag{4.40}$$

subject to the flow budget constraint, given by,

$$P_tC_t + B_t = I_{t-1}B_{t-1} + W_tN_t - T_t, \tag{4.41}$$

where $1/\sigma$ is the elasticity of intertemporal substitution, $1/\eta$ is the Frisch elasticity of labor supply, C_t is the quantity consumed of the single good at price P_t, N_t is hours worked, and B_t is a one-period risk-free nominal bond. Each bond purchased at \$1 pays \$$I_t$ at maturity. T_t is lump-sum taxes, expressed nominally, and W_t is the nominal wage rate. In addition to (4.41), assume the household is subject to solvency constraint that prevents it from engaging in Ponzi-type schemes.

The household's optimality conditions are given by

$$\frac{W_t}{P_t} = C_t^\sigma N_t^\eta, \tag{4.42}$$

$$1 = \beta E_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{I_t}{\Pi_{t+1}}\right\}, \tag{4.43}$$

where $\Pi_{t+1} \equiv P_{t+1}/P_t$ is the gross inflation rate. Consider a log-linear approximation of the model where lower case letters are the natural logs of the corresponding variables (i.e., $x_t = \log X_t$). Log-linearizing (4.42) and (4.43) implies

$$w_t - p_t = \sigma c_t + \eta n_t. \tag{4.44}$$

$$c_t = E_tc_{t+1} - \sigma^{-1}(i_t - E_t\pi_{t+1} - \rho), \tag{4.45}$$

where $\rho \equiv -\log\beta$ is the rate of discount.

The representative firm maximizes profits, $P_tY_t - W_tN_t$, subject to its production function, given by,

$$Y_t = A_tN_t^{1-\alpha}, \tag{4.46}$$

where A_t is the level of technology and $a_t = \log A_t$ evolves exogenously according to some stochastic process. The optimality condition implies

$$\frac{W_t}{P_t} = (1-\alpha)A_tN_t^{-\alpha}.$$

After log-linearizing we obtain

$$w_t - p_t = a_t - \alpha n_t + \log(1-\alpha), \tag{4.47}$$

which can be interpreted as a labor demand schedule.

Without investment or government spending, the aggregate resource constraint is simply $c_t = y_t$. Combining (4.44), (4.47), and the log-linear production function, $y_t = a_t + (1-\alpha)n_t$, implies

$$y_t = \frac{1+\eta}{(1-\alpha)\sigma+\eta+\alpha}a_t + \frac{(1-\alpha)\log(1-\alpha)}{(1-\alpha)\sigma+\eta+\alpha} \equiv \psi_{ya}a_t + \vartheta_y. \tag{4.48}$$

Plugging the solution into the production function implies

$$n_t = \frac{1-\sigma}{(1-\alpha)\sigma+\eta+\alpha}a_t + \frac{\log(1-\alpha)}{(1-\alpha)\sigma+\eta+\alpha} \equiv \psi_{na}a_t + \vartheta_n. \tag{4.49}$$

Moreover, given (4.48), we can use (4.45) to determine the real interest rate, $r_t = i_t - E_t\pi_{t+1}$, as

$$\begin{aligned} r_t &= \rho + \sigma(E_t y_{t+1} - y_t) \\ &= \rho + \sigma\psi_{ya}(E_t a_{t+1} - a_t). \end{aligned} \tag{4.50}$$

Finally, using (4.47) and (4.49), the equilibrium real wage is given by

$$w_t - p_t = \frac{\sigma + \eta}{(1-\alpha)\sigma + \eta + \alpha} a_t + \frac{(\sigma(1-\alpha) + \eta)\log(1-\alpha)}{(1-\alpha)\sigma + \eta + \alpha} \equiv \psi_{wa} a_t + \vartheta_w.$$

The equilibrium values of output, employment, the real interest rate, and the real wage are all determined independently on monetary policy. Money is neutral with respect to real variables. Real variables are only affected by real shocks (in this case, technology, but preference or government spending shocks are other examples).

Notice that output and the real wage unambiguously rise in response to a technology shock, with the size of the increases equal to ψ_{ya} and ψ_{wa}. However, the change in employment following a technology shock is dependent on whether $\sigma \lesseqgtr 1$. When $\sigma < 1$ agents are more willing to substitute consumption across time. Hence, the substitution effect from a higher real wage (which increases labor supply) dominates the income effect (which increases consumption and decreases labor supply as the marginal utility of consumption falls), leading to an increase in employment. When $\sigma = 1$ (log utility in consumption), employment is unaffected, as income and substitution effect cancel each other out. The response of the real interest rate is dependent on the stochastic process for technology. If the technology shock is transitory, then $a_t > E_t a_{t+1}$, and the real interest rate will fall (i.e., the technology shock will have a small effect on future output and consumption and therefore have little effect on the real interest rare).

Nominal variables, such as inflation and the nominal interest rate, are pinned down by monetary policy (assuming lump-sum taxes that passively adjust to clear the government budget constraint). Assume monetary policy follows a simple Taylor rule, given by,

$$i_t = \rho + \phi_\pi \pi_t,$$

where $\phi_\pi > 1$. Combining the Taylor rule with the Fisher equation, $r_t = i_t - E_t\pi_{t+1}$, implies

$$\phi_\pi \pi_t = E_t \pi_{t+1} + \hat{r}_t,$$

where $\hat{r}_t \equiv r_t - \rho$. Iterating forward, the solution for inflation is given by

$$\pi_t = \sum_{k=0}^{\infty} \phi_\pi^{-(k+1)} E_t \hat{r}_{t+k}.$$

This equation determines inflation (and hence the price level) as a function of the real interest rate, which is a function of the fundamentals, as shown in (4.50). Hence, monetary policy pins down nominal variables. However, since monetary policy has no effect on real variables, particularly consumption and employment, no policy rule is better than the other—a policy rule that generates large fluctuations in prices is no less desirable than one that stabilizes price. Thus, the classical monetary model cannot explain the observed real effects of monetary policy. This is the main motivation for the introduction of nominal frictions.

4.5 Basic New Keynesian Model

This section is based on Galí (2008), chapter 3, and Walsh (2010), chapter 8. In the classical monetary model, there is a very limited role for money. It only serves as a unit of account. We saw that real variables are determined independently of monetary policy, and, hence, monetary policy is neutral. The *New Keynesian* model contains two key new features:

- Nominal rigidities: Imposes constraints on price adjustment. For example, only a fraction of firms can reset their price in any given period [Calvo (1983)]. Alternatively, firms may face some cost of adjusting their prices [Rotemberg (1982)].
- Monopolistic competition: Each firm produces a differentiated good for which it sets the price. In the presence of sticky prices this feature is needed because under perfect competition, any firm with a price slightly higher than the others would be unable to sell anything, and any firm with a price slightly lower than the others would be obliged to sell much more than they can profitably produce.

As a consequence of nominal rigidities, changes in the short-term nominal interest rate are not matched one-for-one with changes in expected inflation, thus leading to variations in real interest rates. This causes changes in consumption and investment (when present), and hence output as firms adjust to changes in demand. *New Keynesian* models deliver short-run non-neutrality of money. In this setting, the economy's response to shocks is generally inefficient. The non-neutrality of monetary policy makes room for potentially welfare-enhancing interventions to minimize distortions.

4.5.1 Households

There is a representative agent who derives utility from a composite consumption good C and also derives disutility from working. The agent provides labor to each firm, where labor supplied to firm i is denoted $N(i)$. Labor is a homogenous good in that the agent does not distinguish between different jobs. Aggregate labor, that is the total time spent working by the agent, is given by $N_t = \int_0^1 N(i)di$. Thus, preferences are still given by (4.40), but now C is a composite consumption good. The slightly modified budget constraint of the household is

$$P_tC_t + B_t = I_{t-1}B_{t-1} + W_tN_t + T_t + D_t, \tag{4.51}$$

where D_t are nominal profits received from firms. Hence, the household's optimality conditions remain unchanged from the classical monetary model.

4.5.2 Firms

The production sector consists of a continuum of monopolistically competitive intermediate goods producers and a representative final goods producer.

Final Goods Sector

The final goods sector is perfectly competitive. A representative firm produces a final good Y using a continuum of intermediate goods labeled $Y(i)$ according to a constant elasticity of substitution (CES) production function, given by,

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{4.52}$$

where $\varepsilon > 1$ measures price elasticity of demand for individual good i. As $\varepsilon \to \infty$, the individual goods become closer substitutes, and consequently individual firms have less market power.

Denote $P_t(i)$ as the price of good i. The representative final goods producing firm chooses intermediate inputs, $Y_t(i)$ to maximize profit, $P_tY_t - \int_0^1 P_t(i)Y_t(i)di$ subject to their production function, given in (4.52). Optimality yields

$$\begin{aligned} P_t(i) &= P_t\frac{\varepsilon}{\varepsilon-1}\left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}di\right]^{\frac{\varepsilon}{\varepsilon-1}-1}\frac{\varepsilon-1}{\varepsilon}Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}-1} \\ &= P_tY_t^{1/\varepsilon}Y_t(i)^{-1/\varepsilon}, \end{aligned}$$

which implies that the demand function for intermediate good i is given by

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}Y_t. \tag{4.53}$$

Substitute the demand function, (4.53) into the final goods production technology, (4.52), to obtain

$$\begin{aligned} Y_t &= \left[\int_0^1\left(Y_t\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\right)^{\frac{\varepsilon-1}{\varepsilon}}di\right]^{\frac{\varepsilon}{\varepsilon-1}} \\ \to 1 &= \left[\int_0^1\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon}di\right]^{\frac{\varepsilon}{\varepsilon-1}} = P_t^{\varepsilon}\left[\int_0^1 P_t(i)^{1-\varepsilon}di\right]^{\frac{\varepsilon}{\varepsilon-1}}. \end{aligned}$$

Thus, the aggregate price level is given by

$$P_t = \left[\int_0^1 P_t(i)^{1-\varepsilon}di\right]^{\frac{1}{1-\varepsilon}}. \tag{4.54}$$

Intermediate Goods Sector

The intermediate goods sector is monopolistically competitive. Each firm produces a differentiated good with identical technologies given by

$$Y_t(i) = A_tN_t(i)^{1-\alpha}, \tag{4.55}$$

where A is a technology shock that is common across all firms. Since each intermediate firm is operating in a monopolistically competitive industry, it has control over its price level. The easiest way to solve the intermediate firms' optimization problem is to break it into two steps. In the first step, we will solve the cost minimization problem, which will give us each firm's marginal cost function. In the second step, we will solve the profit maximization problem, which will give us each firm's price.

Step 1: Each firm chooses its inputs to minimize its real cost for a given level of output. That is, the firm solves

$$\begin{aligned} &\min_{N_t(i)} \frac{W_t}{P_t}N_t(i) \\ &\text{subject to } Y_t(i) = A_tN_t(i)^{1-\alpha}. \end{aligned}$$

The Lagrange multiplier represents real marginal costs (i.e., the shadow price of an additional unit of output). Thus, the optimality condition is given by

$$\begin{aligned}\frac{W_t}{P_t} &= \mathrm{MC}_t(i)(1-\alpha)A_tN_t(i)^{-\alpha} = \mathrm{MC}_t(i)(1-\alpha)\frac{Y_t(i)}{N_t(i)}.\\ &= \mathrm{MC}_t(i)(1-\alpha)A_t(Y_t(i)/A_t)^{-\alpha/(1-\alpha)}\\ &= \mathrm{MC}_t(i)(1-\alpha)A_t^{1/(1-\alpha)}(Y_t(i))^{-\alpha/(1-\alpha)}\\ &= \mathrm{MC}_t(i)(1-\alpha)A_t^{1/(1-\alpha)}(P_t(i)/P_t)^{-\alpha\tilde{\varepsilon}}(Y_t)^{-\alpha/(1-\alpha)},\end{aligned}$$

where $\tilde{\varepsilon} \equiv \varepsilon/(1-\alpha)$. Note that under constant returns to scale ($\alpha = 0$), marginal costs are independent of the intermediate good, but with decreasing returns to scale ($\alpha > 0$) marginal costs are dependent on the good. The average real marginal cost function satisfies

$$\frac{W_t}{P_t} = \mathrm{MC}_t(1-\alpha)A_tN_t^{-\alpha}. \tag{4.56}$$

The ratio of firm i's marginal costs to the average marginal costs is given by

$$\mathrm{MC}_t(i) = \mathrm{MC}_t(P_t(i)/P_t)^{-\alpha\tilde{\varepsilon}}.$$

Thus, total costs can be written as

$$\begin{aligned}TC_t(i)/P_t = W_tN_t(i)/P_t &= (1-\alpha)\,\mathrm{MC}_t(i)Y_t(i)\\ &= (1-\alpha)\,\mathrm{MC}_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}Y_t\\ &= (1-\alpha)\,\mathrm{MC}_t\left(\frac{P_t(i)}{P_t}\right)^{-\tilde{\varepsilon}}Y_t.\end{aligned} \tag{4.57}$$

Step 2 Instead of allowing firms to change their price each period, assume prices are sticky according to the discrete time version of Calvo (1983). Each firm may reset its price with probability $1-\omega$ in any given period (a measure $1-\omega$ of the producers can adjust prices and a fraction ω keep prices unchanged). The probability mass function is given by

$$f(k) = (1-\omega)\omega^{k-1}, \quad k = 1, 2, 3, \ldots$$

where k represents number of periods until prices may be reset, and $f(k)$ is the probability of not being able to reoptimize prices until period k. The average duration of a price is given by[1]

$$E[k] = \sum_{k=1}^{\infty} kf(k) = (1-\omega)\sum_{k=1}^{\infty} k\omega^{k-1} = \frac{1}{1-\omega}.$$

Thus, ω measures the degree of price stickiness.

While individual firms produce differentiated products, they all have the same production technology and face demand curves with constant and equal demand elasticities. In other words, they are essentially identical except that they may have set their current price at different dates in the past. However, all firms that can adjust their price in period t face the same problem, so all adjusting firms will set the same price. Firms that are able to adjust their prices in period t, choose their

[1] To evaluate this sum simply differentiate the definition $\sum_{i=0}^{\infty} x^i = 1/(1-x)$.

price level, P_t^*, to maximize the discounted present value of future real profits. Using (4.53) and (4.57), real profits for firm i in period t are given by

$$\frac{D_t(i)}{P_t} = \frac{P_t(i)}{P_t}Y_t(i) - \frac{TC_t(i)}{P_t} = \left[\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon} - (1-\alpha)\,\mathrm{MC}_t\left(\frac{P_t(i)}{P_t}\right)^{-\tilde{\varepsilon}}\right]Y_t$$

Thus, firms that are able to reset their prices choose P_t^* to maximize

$$E_t\sum_{s=t}^{\infty}\omega^{s-t}Q_{t,s}\left[\left(\frac{P_t^*}{P_s}\right)^{1-\varepsilon} - (1-\alpha)\,\mathrm{MC}_s\left(\frac{P_t^*}{P_s}\right)^{-\tilde{\varepsilon}}\right]Y_s,$$

where $Q_{t,s}$ is the stochastic discount factor between periods t and s, defined as

$$Q_{t,s} \equiv \begin{cases} \prod_{j=t}^{s-1}\beta\left(\frac{C_j}{C_{j+1}}\right)^{\sigma} = \beta^{s-t}\left(\frac{C_t}{C_s}\right)^{\sigma} & s>t \\ 1 & s=t \end{cases}$$

The optimality condition implies

$$\begin{aligned}
&E_t\sum_{s=t}^{\infty}\omega^{s-t}Q_{t,s}\left\{(1-\varepsilon)\left(\frac{P_t^*}{P_s}\right)^{-\varepsilon}\frac{Y_s}{P_s} + \varepsilon\,\mathrm{MC}_s\left(\frac{P_t^*}{P_s}\right)^{-\tilde{\varepsilon}-1}\frac{Y_s}{P_s}\right\} = 0 \\
&\rightarrow (\varepsilon-1)E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_t^*}{P_s}\right)^{1-\varepsilon}Y_s = \varepsilon E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_t^*}{P_s}\right)^{-\tilde{\varepsilon}}\mathrm{MC}_s\,Y_s \\
&\rightarrow 1 = \frac{\varepsilon}{\varepsilon-1}\frac{E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_t^*}{P_s}\right)^{-\tilde{\varepsilon}}\mathrm{MC}_s\,Y_s}{E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_t^*}{P_s}\right)^{1-\varepsilon}Y_s} \\
&\rightarrow \left(\frac{P_t^*}{P_t}\right)^{1/\Theta} = \frac{\varepsilon}{\varepsilon-1}\frac{E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_s}{P_t}\right)^{\tilde{\varepsilon}}\mathrm{MC}_s\,Y_s}{E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_s}{P_t}\right)^{\varepsilon-1}Y_s},
\end{aligned}$$

which can be written more compactly as

$$\frac{P_t^*}{P_t} = \left[\left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{X_{1,t}}{X_{2,t}}\right]^{\Theta} = \left[\mathcal{M}\frac{X_{1,t}}{X_{2,t}}\right]^{\Theta}, \tag{4.58}$$

where $\mathcal{M} \equiv \varepsilon/(\varepsilon-1)$ is the markup factor in the absence of nominal frictions and

$$\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon} \leq 1.$$

The numerator in the intermediate firm pricing equation can be written as

$$\begin{aligned}
X_{1,t} &= E_t\sum_{s=t}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_s}{P_t}\right)^{\tilde{\varepsilon}}\mathrm{MC}_s\,Y_s \\
&= C_t^{-\sigma}\,\mathrm{MC}_t\,Y_t + E_t\left\{\sum_{s=t+1}^{\infty}(\beta\omega)^{s-t}C_s^{-\sigma}\left(\frac{P_s}{P_t}\right)^{\tilde{\varepsilon}}\mathrm{MC}_s\,Y_s\right\} \\
&= C_t^{-\sigma}\,\mathrm{MC}_t\,Y_t + \beta\omega E_t\left\{\left(\frac{P_{t+1}}{P_t}\right)^{\tilde{\varepsilon}}E_{t+1}\left[\sum_{s=t+1}^{\infty}(\beta\omega)^{s-(t+1)}C_s^{-\sigma}\left(\frac{P_s}{P_{t+1}}\right)^{\tilde{\varepsilon}}\mathrm{MC}_s\,Y_s\right]\right\}
\end{aligned}$$

Following the same steps we can rewrite the denominator. Thus, X_1 and X_2 have recursive representations, given by,

$$X_{1,t} = C_t^{-\sigma}\,\mathrm{MC}_t\,Y_t + \beta\omega E_t\left\{\Pi_{t+1}^{\tilde{\varepsilon}} X_{1,t+1}\right\}, \tag{4.59}$$
$$X_{2,t} = C_t^{-\sigma} Y_t + \beta\omega E_t\left\{\Pi_{t+1}^{\varepsilon-1} X_{2,t+1}\right\}, \tag{4.60}$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. In the special case where all firms are able to adjust their price every period ($\omega = 0$), the firm's pricing equation reduces to

$$\frac{P_t^*}{P_t} = [\mathcal{M}\,\mathrm{MC}_t]^{\Theta} = [\mathcal{M}\,\mathrm{MC}_t(i)]^{\Theta} \to \mathrm{MC}_t(i) = 1/\mathcal{M}. \tag{4.61}$$

Thus, each firm sets its price, P_t^* equal to a markup, $\mathcal{M} > 1$ over its nominal marginal cost, $P_t\,\mathrm{MC}_t(i)$. When prices are flexible, all firms charge the same price. In this case $P_t^* = P_t$. Using (4.42) and (4.56)

$$\frac{W_t}{P_t} = \frac{1-\alpha}{\mathcal{M}} A_t N_t^{-\alpha} = C_t^{\sigma} N_t^{\eta}.$$

Goods market clearing and the production function imply, $C_t = Y_t$ and $N_t = (Y_t/A_t)^{1/(1-\alpha)}$. Thus, letting Y^n denote the natural rate of output (equilibrium output under flexible prices), we obtain

$$Y_t^n = \left[A_t^{1+\eta}\left(\frac{1-\alpha}{\mathcal{M}}\right)^{1-\alpha}\right]^{\frac{1}{\sigma(1-\alpha)+\eta+\alpha}}. \tag{4.62}$$

When prices are flexible, output is a function of the productivity shock, reflecting the fact that in the absence of sticky prices, the new Keynesian model reduces to the real business cycle model.

4.5.3 Aggregate Price Dynamics

Finally, we can express the aggregate price level, P_t, in terms of the last period's aggregate price index and the optimal chosen price level at time t. To do this, note that the aggregate price level, (4.54), can be written as

$$P_t^{1-\varepsilon} = \int_0^1 P_t(i)^{1-\varepsilon} di.$$

First, we need to integrate across the individual price choices of the firms. Since the price signal follows a Bernoulli process and the process is independent across firms, a measure of exactly $(1-\omega)$ firms will get to re-optimize their prices today. Each firm i will choose P_t^*. The remaining firms, of measure ω, cannot change prices at time t and keep the same price they had before. Notice the same situation happened each period in the past, e.g. in period $t-1$, $(1-\omega)$ firms chose P_{t-1}^* while ω firms kept their price the same. Using this, we can write the aggregate price index, (4.54) as

$$\begin{aligned} P_t^{1-\varepsilon} &= (1-\omega)(P_t^*)^{1-\varepsilon} + \omega(1-\omega)(P_{t-1}^*)^{1-\varepsilon} + \omega^2(1-\omega)(P_{t-2}^*)^{1-\varepsilon} + \cdots \\ &= (1-\omega)(P_t^*)^{1-\varepsilon} + \omega P_{t-1}^{1-\varepsilon} \end{aligned}$$

Dividing by $P_t^{1-\varepsilon}$ and solving for P_t^*/P_t implies

$$\left[\mathcal{M}\frac{X_{1,t}}{X_{2,t}}\right]^{\Theta} = \left[\frac{1-\omega\Pi_t^{\varepsilon-1}}{1-\omega}\right]^{\frac{1}{1-\varepsilon}}, \tag{4.63}$$

when we impose (4.58). Using (4.53) and (4.55), we obtain

$$\begin{aligned} N_t &= \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{1/(1-\alpha)} di \\ &= \left(\frac{Y_t}{A_t}\right)^{1/(1-\alpha)} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon/(1-\alpha)} di \end{aligned}$$

Thus, the aggregate production function is given by

$$Y_t \Delta_t^{1-\alpha} = A_t N_t^{1-\alpha}, \tag{4.64}$$

where $\Delta_t \equiv \int_0^1 (P_t(i)/P_t)^{-\tilde{\varepsilon}} di$ is price dispersion. Similar to the aggregate price level, we can write price dispersion as

$$\begin{aligned} \Delta_t &\equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\tilde{\varepsilon}} di \\ &= (1-\omega)\left(\frac{P_t^*}{P_t}\right)^{-\tilde{\varepsilon}} + \omega(1-\omega)\left(\frac{P_{t-1}^*}{P_t}\right)^{-\tilde{\varepsilon}} + \omega^2(1-\omega)\left(\frac{P_{t-2}^*}{P_t}\right)^{-\tilde{\varepsilon}} + \cdots \\ &= (1-\omega)\left(\frac{P_t^*}{P_t}\right)^{-\tilde{\varepsilon}} + \omega\left(\frac{P_t}{P_{t-1}}\right)^{\tilde{\varepsilon}}\left[(1-\omega)\left(\frac{P_{t-1}^*}{P_{t-1}}\right)^{-\tilde{\varepsilon}} + \omega(1-\omega)\left(\frac{P_{t-2}^*}{P_{t-1}}\right)^{-\tilde{\varepsilon}} + \cdots\right] \end{aligned}$$

Thus, written recursively, price dispersion is given by

$$\Delta_t = (1-\omega)\left(\mathcal{M}\frac{X_{1,t}}{X_{2,t}}\right)^{-\tilde{\varepsilon}\Theta} + \omega \Pi_t^{\tilde{\varepsilon}} \Delta_{t-1} \tag{4.65}$$

Applying a first-order Taylor approximation around a zero-inflation steady state shows that price dispersion, $\delta_t = \log \Delta_t$, is zero. In order to capture any welfare effects related to price dispersion, it is necessary to work with higher order approximations (second order for example). To see this, rearrange the nominal price index, (4.54) and apply first-order Maclaurin expansion to obtain

$$\begin{aligned} 1 = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon} di &= \int_0^1 \exp\left\{\ln\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon}\right\} di \\ &= \int_0^1 \exp\{(1-\varepsilon)(p_t(i)-p_t)\} \\ &\approx \int_0^1 \{1+(1-\varepsilon)(p_t(i)-p_t)\} di \\ &= 1 + (1-\varepsilon)\int_0^1 (p_t(i)-p_t) di, \end{aligned}$$

which implies that $p_t \approx \int_0^1 p_t(i)di$. Thus,

$$\begin{aligned}\Delta_t &= \int_0^1 (P_t(i)/P_t)^{-\tilde{\varepsilon}}\,di = \int_0^1 \exp\left\{\ln\left(\frac{P_t(i)}{P_t}\right)^{-\tilde{\varepsilon}}\right\}di \\ &= \int_0^1 \exp\{-\tilde{\varepsilon}(p_t(i)-p_t)\} \\ &\approx \int_0^1 \{1-\tilde{\varepsilon}(p_t(i)-p_t)\}di \\ &= 1-\tilde{\varepsilon}\int_0^1 (p_t(i)-p_t)di \\ &= 1.\end{aligned}$$

Thus, a first-order approximation implies price dispersion, $\delta_t = 0$.

4.5.4 Equilibrium

Market clearing in the goods market requires $Y_t(i) = C_t(i)$ for all $i \in [0,1]$ and all t. Letting aggregate consumption be defined as $C_t \equiv (\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}}di)^{\frac{\varepsilon}{\varepsilon-1}}$ it follows that

$$C_t = Y_t \tag{4.66}$$

To summarize, (4.42), (4.43), (4.56), (4.59), (4.60), (4.63), (4.64), (4.65), and (4.66) represent a system in W_t/P_t, C_t, N_t, I_t, Π_t, $X_{1,t}$, $X_{2,t}$, Y_t, MC_t, and Δ_t (10 variables) that can be combined with a specification of monetary policy to determine the economy's equilibrium (10 equations). To this system, later sections will add exogenous processes that govern technology and monetary policy shocks.

One reason for the popularity of the new Keynesian model is that it allows for a simple linear representation in terms of a Phillips curve and an output and real interest rate relationship that corresponds to the IS curve often taught in undergraduate economics courses. The log-linear equilibrium system of equations is given by

$$w_t - p_t = \sigma c_t + \eta n_t \tag{4.67}$$
$$c_t = E_t c_{t+1} - \sigma^{-1}(i_t - E_t\pi_{t+1} - \rho) \tag{4.68}$$
$$w_t - p_t = \mathrm{mc}_t + \log(1-\alpha) + a_t - \alpha n_t \tag{4.69}$$
$$\hat{x}_{1,t} = (1-\beta\omega\Pi^{\tilde{\varepsilon}})(-\sigma\hat{c}_t + \widehat{\mathrm{mc}}_t + \hat{y}_t) + \beta\omega\Pi^{\tilde{\varepsilon}}E_t\{\hat{x}_{1,t+1} + \tilde{\varepsilon}\hat{\pi}_{t+1}\} \tag{4.70}$$
$$\hat{x}_{2,t} = (1-\beta\omega\Pi^{\varepsilon-1})(-\sigma\hat{c}_t + \hat{y}_t) + \beta\omega\Pi^{\varepsilon-1}E_t\{\hat{x}_{2,t+1} + (\varepsilon-1)\hat{\pi}_{t+1}\} \tag{4.71}$$
$$\omega\Pi^{\varepsilon-1}\hat{\pi}_t = \Theta(1-\omega\Pi^{\varepsilon-1})(\hat{x}_{1,t} - \hat{x}_{2,t}) \tag{4.72}$$
$$\hat{y}_t = a_t + (1-\alpha)\hat{n}_t \tag{4.73}$$
$$\hat{c}_t = \hat{y}_t \tag{4.74}$$

where a hat denotes log deviations from the deterministic steady state. If we assume a zero inflation steady state ($\Pi = 1$), the equilibrium system simplifies to the familiar new Keynesian Phillips curve. To see this, subtract (4.71) from (4.70) to obtain

$$\hat{x}_{1,t} - \hat{x}_{2,t} = (1-\beta\omega)\widehat{\mathrm{mc}}_t + \beta\omega(E_t\hat{x}_{1,t+1} - E_t\hat{x}_{2,t+1} + \Theta^{-1}E_t\pi_{t+1})$$

Then use (4.72) to substitute for $\hat{x}_1 - \hat{x}_2$ and simplify to obtain

$$\pi_t = \beta E_t \pi_{t+1} + \lambda \widehat{\text{mc}}_t, \tag{4.75}$$

where

$$\lambda \equiv \frac{(1-\omega)(1-\beta\omega)}{\omega}\Theta = \frac{(1-\omega)(1-\beta\omega)}{\omega}\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$$

and

$$\frac{\partial\lambda}{\partial\alpha} = \frac{(1-\omega)(1-\beta\omega)}{\omega}\left[\frac{(1-\alpha+\alpha\varepsilon)(-1)-(1-\alpha)(-1+\varepsilon)}{(1-\alpha+\alpha\varepsilon)^2}\right] = -\frac{\varepsilon(1-\omega)(1-\beta\omega)}{\omega(1-\alpha+\alpha\varepsilon)^2} < 0$$
$$\frac{\partial\lambda}{\partial\omega} = \Theta\left[\frac{\omega[(1-\omega)(-\beta)+(1-\beta\omega)(-1)]-(1-\omega)(1-\beta\omega)}{\omega^2}\right] = \Theta\left(\frac{\beta\omega^2-1}{\omega^2}\right) < 0.$$

An increase in β means that firms give more weight to expected future profits. As a consequence, λ declines and inflation is less sensitive to current marginal costs. Increased price rigidity (a rise in ω) reduces λ; With opportunities to adjust prices arriving less frequently, the firm places less weight on current marginal costs when it does adjust its price. Solving (4.75) forward gives

$$\pi_t = \lambda\sum_{k=0}^{\infty}\beta^k E_t \widehat{\text{mc}}_{t+k},$$

which shows that inflation is a function of the present discounted value of current and future real marginal costs. Equation (4.75) is often referred to as the new Keynesian Phillips curve. It implies that the inflation process is forward looking, with current inflation a function of expected inflation. When a firm sets its price, it must be concerned with inflation in the future because it may be unable to adjust its price for several periods.

The Phillips Curve and the Output Gap

Equation (4.75) implies that inflation depends on real marginal cost and not directly on a measure of the gap between actual output and some measure of potential output or on a measure of unemployment relative to its natural rate, as is typical in traditional Phillips curves. However, real marginal costs can be related to an output gap measure. We now seek a relation between the economy's average real marginal cost and a measure of economic activity. Using (4.67), (4.69), and (4.73) real marginal costs in log terms can be written as

$$\begin{aligned}
\text{mc}_t &= (\sigma y_t + \eta n_t) - (a_t - \alpha n_t) - \log(1-\alpha) \\
&= \left(\sigma y_t + \frac{\eta}{1-\alpha}(y_t - a_t)\right) - \left(a_t - \frac{\alpha}{1-\alpha}(y_t - a_t)\right) - \log(1-\alpha) \\
&= \left(\sigma + \frac{\eta+\alpha}{1-\alpha}\right) y_t - \frac{1+\eta}{1-\alpha} a_t - \log(1-\alpha).
\end{aligned}$$

Define y_t^n as the log of natural rate of output (the equilibrium level of output under flexible prices). In this case, using (4.61), the above condition can be rewritten as

$$-\mu = \left(\sigma + \frac{\eta+\alpha}{1-\alpha}\right) y_t^n - \frac{1+\eta}{1-\alpha} a_t - \log(1-\alpha),$$

which implies that

$$y_t^n = \psi_{ya}^n a_t + \mathcal{V}_y^n, \tag{4.76}$$

where

$$\psi_{ya}^n \equiv \frac{1+\eta}{\sigma(1-\alpha)+\eta+\alpha}, \qquad \mathcal{V}_y^n \equiv -\frac{(1-\alpha)(\mu-\log(1-\alpha))}{\sigma(1-\alpha)+\eta+\alpha},$$

which is simply the log of (4.62). Notice that when $\mu = 0$ (perfect competition), the natural level of output corresponds to level of output derived in the classical monetary model, given in (4.48). A higher markup lowers output. The difference of these equations then gives

$$\widehat{\mathrm{mc}}_t = \mathrm{mc}_t + \mu = \left(\sigma + \frac{\eta+\alpha}{1-\alpha}\right)(y_t - y_t^n), \tag{4.77}$$

which implies that the log deviation of real marginal costs is proportional to the deviation of output from the flexible price output (output gap). Combining equation (4.75) and (4.77) yields

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \tilde{y}_t, \tag{4.78}$$

where $\tilde{y}_t \equiv y_t - y_t^n$ measures the output gap and $\kappa \equiv \lambda\left(\sigma + \frac{\eta+\alpha}{1-\alpha}\right)$. This is the first key equation in the linear equilibrium system. To derive the second key equation, rewrite the equilibrium consumption Euler equation, (4.68), to obtain

$$\begin{aligned} y_t - y_t^n &= E_t[y_{t+1} - y_{t+1}^n] + E_t y_{t+1}^n - y_t^n - \frac{1}{\sigma}(i_t - E_t\pi_{t+1} - \rho) \\ \rightarrow \tilde{y}_t &= E_t \tilde{y}_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - \rho - \sigma E_t \Delta y_{t+1}^n). \end{aligned}$$

Thus, the IS equation is given by

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \sigma^{-1}(i_t - E_t\pi_{t+1} - r_t^n), \tag{4.79}$$

where $\Delta y_{t+1}^n \equiv E_t y_{t+1}^n - y_t^n$, and

$$\begin{aligned} r_t^n &\equiv \rho + \sigma E_t \Delta y_{t+1}^n \\ &= \rho + \sigma \psi_{ya}^n E_t \Delta a_{t+1}, \end{aligned} \tag{4.80}$$

measures the natural rate of interest. If $\lim_{T\to\infty} E_t \tilde{y}_{t+T} = 0$, this condition can be written as

$$\tilde{y}_t = -\frac{1}{\sigma} E_t \sum_{k=0}^{\infty} (r_{t+k} - r_{t+k}^n),$$

where $r_t \equiv i_t - E_t \pi_{t+1}$. This says that the output gap is proportional to the sum of current and expected future deviations of the real interest rate from the natural rate.

Equations (4.78) and (4.79), together with an equilibrium process form the natural rate of interest, for the non-policy block of the equilibrium system.

Equilibrium Under a Simple Interest Rate Rule

To close the model, postulate a simple interest rate rule of the form

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \upsilon_t, \tag{4.81}$$

where υ_t is exogenous with zero mean. This type of policy rule is called a *Taylor rule* (Taylor, 1993), and variants of it have been shown to provide a reasonable empirical description of the policy behavior of many central banks. Combining (4.79) and (4.81), one obtains

$$\begin{aligned}
\tilde{y}_t &= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left[\rho + \phi_\pi(\beta E_t\pi_{t+1} + \kappa\tilde{y}_t) + \phi_y\tilde{y}_t + \upsilon_t - E_t\pi_{t+1} - r_t^n\right],\\
\rightarrow \left(1 + \frac{\phi_y + \kappa\phi_\pi}{\sigma}\right)\tilde{y}_t &= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\beta\phi_\pi - 1)E_t\pi_{t+1} - \frac{1}{\sigma}(\rho + \upsilon_t - r_t^n),\\
\rightarrow \tilde{y}_t &= \Omega\sigma E_t\tilde{y}_{t+1} + \Omega(1 - \beta\phi_\pi)E_t\pi_{t+1} + \Omega(\hat{r}_t^n - \upsilon_t),
\end{aligned}$$

where $\hat{r}_t^n = r_t^n - \rho$ and $\Omega = 1/(\sigma + \phi_y + \kappa\phi_\pi)$. Using the above result, (4.78) can be written as

$$\begin{aligned}
\pi_t &= \beta E_t\pi_{t+1} + \kappa[\Omega\sigma E_t\tilde{y}_{t+1} + \Omega(1 - \beta\phi_\pi)E_t\pi_{t+1} + \Omega(\hat{r}_t^n - \upsilon_t)]\\
&= \Omega\sigma\kappa E_t\tilde{y}_{t+1} + [\Omega\kappa(1 - \beta\phi_\pi) + \beta]E_t\pi_{t+1} + \Omega\kappa(\hat{r}_t^n - \upsilon_t)\\
&= \Omega\sigma\kappa E_t\tilde{y}_{t+1} + \Omega[\kappa(1 - \beta\phi_\pi) + \beta(\sigma + \phi_y + \kappa\phi_\pi)]E_t\pi_{t+1} + \Omega\kappa(\hat{r}_t^n - \upsilon_t)\\
&= \Omega\sigma\kappa E_t\tilde{y}_{t+1} + \Omega[\kappa + \beta(\sigma + \phi_y)]E_t\pi_{t+1} + \Omega\kappa(\hat{r}_t^n - \upsilon_t).
\end{aligned}$$

Thus, using the above results, we have the following system

$$\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix} = \mathbf{A}_T\begin{bmatrix}E_t\tilde{y}_{t+1}\\ E_t\pi_{t+1}\end{bmatrix} + \mathbf{B}_T(\hat{r}_t^n - \upsilon_t), \tag{4.82}$$

where

$$\mathbf{A}_T \equiv \Omega\begin{bmatrix}\sigma & 1 - \beta\phi_\pi\\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y)\end{bmatrix}, \qquad \mathbf{B}_T \equiv \Omega\begin{bmatrix}1\\ \kappa\end{bmatrix}.$$

The solution is locally stable if and only if both eigenvalues of $\mathbf{A}_T$ lie inside the unit circle. Given $\phi_\pi \geq 0$ and $\phi_y \geq 0$ and necessary and sufficient condition for determinacy is

$$\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y > 0. \tag{4.83}$$

4.5.5 Effects of a Monetary Policy Shock

Assume the determinacy condition, (4.83), holds. Let υ_t (the exogenous component of the nominal interest rate) follow an AR(1) process, given by,

$$\upsilon_t = \rho_\upsilon \upsilon_{t-1} + \varepsilon_t^\upsilon,$$

where υ is a zero mean white noise process. Since the natural rate of interest, $\hat{r}_t^n$, is unaffected by the monetary policy shock, we will set $\hat{r}_t^n = 0$ for convenience. In order to solve the model, posit

$$\begin{aligned}
\tilde{y}_t &= a\upsilon_t,\\
\pi_t &= b\upsilon_t,
\end{aligned}$$

where a and b are unknown coefficients. Then, from the AR(1) process,

$$\begin{aligned} E_t\tilde{y}_{t+1} &= a\rho_\upsilon \upsilon_t \\ E_t\pi_{t+1} &= b\rho_\upsilon \upsilon_t. \end{aligned}$$

Plugging our guess into the above system, (4.82), gives

$$\begin{aligned} a\upsilon_t &= \Omega\sigma\rho_\upsilon\upsilon_t + \Omega(1-\beta\phi_\pi)b\rho_\upsilon\upsilon_t - \Omega\upsilon_t \\ b\upsilon_t &= \Omega\sigma\kappa a\rho_\upsilon\upsilon_t + \Omega[\kappa+\beta(\sigma+\phi_y)]b\rho_\upsilon\upsilon_t - \Omega\kappa\upsilon_t. \end{aligned}$$

Equating coefficients on υ_t then yields

$$\underbrace{\begin{bmatrix} 1-\Omega\sigma\rho_\upsilon & -\Omega\rho_\upsilon(1-\beta\phi_\pi) \\ -\Omega\sigma\kappa\rho_\upsilon & 1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)] \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\Omega \\ -\Omega\kappa \end{bmatrix}.$$

Thus,

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 1-\Omega\sigma\rho_\upsilon & -\Omega\rho_\upsilon(1-\beta\phi_\pi) \\ -\Omega\sigma\kappa\rho_\upsilon & 1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)] \end{bmatrix}^{-1} \begin{bmatrix} -\Omega \\ -\Omega\kappa \end{bmatrix} \\ &= \frac{1}{\det A} \begin{bmatrix} 1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)] & \Omega\rho_\upsilon(1-\beta\phi_\pi) \\ \Omega\sigma\kappa\rho_\upsilon & 1-\Omega\sigma\rho_\upsilon \end{bmatrix} \begin{bmatrix} -\Omega \\ -\Omega\kappa \end{bmatrix} \\ &= \frac{1}{\det A} \begin{bmatrix} -\Omega\{1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)]\} - \Omega^2\rho_\upsilon\kappa(1-\beta\phi_\pi) \\ -\Omega^2\sigma\kappa\rho_\upsilon - \Omega\kappa(1-\Omega\sigma\rho_\upsilon) \end{bmatrix} \\ &= \frac{1}{\det A} \begin{bmatrix} -\Omega + \Omega^2\rho_\upsilon\beta[\sigma+\phi_y+\kappa\phi_\pi] \\ -\Omega\kappa \end{bmatrix} \\ &= \frac{1}{\det A} \begin{bmatrix} -\Omega(1-\beta\rho_\upsilon) \\ -\Omega\kappa \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \det A &= [1-\Omega\sigma\rho_\upsilon][1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)]] - \Omega^2\rho_\upsilon^2\sigma\kappa(1-\beta\phi_\pi) \\ &= 1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)] - \Omega\sigma\rho_\upsilon + \Omega^2\rho_\upsilon^2\sigma\beta(\sigma+\phi_y+\kappa\phi_\pi) \\ &= 1-\Omega\rho_\upsilon[\kappa+\beta(\sigma+\phi_y)] - \Omega\sigma\rho_\upsilon + \Omega\rho_\upsilon^2\sigma\beta \\ &= \Omega(\sigma+\phi_y+\kappa\phi_\pi) - \Omega\beta\rho_\upsilon[\sigma(1-\rho_\upsilon)+\phi_y] - \Omega\sigma\rho_\upsilon - \Omega\kappa\rho_\upsilon \\ &= \Omega[\sigma(1-\rho_\upsilon)+\phi_y] - \Omega\beta\rho_\upsilon[\sigma(1-\rho_\upsilon)+\phi_y] + \Omega\kappa(\phi_\pi-\rho_\upsilon) \\ &= \Omega[(1-\beta\rho_\upsilon)[\sigma(1-\rho_\upsilon)+\phi_y] + \kappa(\phi_\pi-\rho_\upsilon)]. \end{aligned}$$

Thus, the model solution is given by

$$\tilde{y}_t = \psi_{y\upsilon}\upsilon_t = -(1-\beta\rho_\upsilon)\Lambda_\upsilon\upsilon_t \tag{4.84}$$

$$\pi_t = \psi_{\pi\upsilon}\upsilon_t = -\kappa\Lambda_\upsilon\upsilon_t, \tag{4.85}$$

where

$$\Lambda_\upsilon \equiv \frac{1}{(1-\beta\rho_\upsilon)[\sigma(1-\rho_\upsilon)+\phi_y] + \kappa(\phi_\pi-\rho_\upsilon)} > 0.$$

When there is a monetary policy contraction (υ_t rises), the output gap ($\tilde{y}_t$) and inflation π_t falls. Given that $r_t = i_t - E_t\pi_{t+1}$, we can use (4.79) and the solution for the output gap, (4.84), to obtain

$$\hat{r}_t = \sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon)\Lambda_\upsilon\upsilon_t.$$

Hence the real interest rate unambiguously rises following a monetary contraction. The response of the nominal interest rate reflects the direct effect of the endogenous response of inflation and the output gap. It is given by

$$\hat{i}_t = \hat{r}_t + E_t\pi_{t+1} = [\sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon) - \rho_\upsilon\kappa]\Lambda_\upsilon\upsilon_t.$$

Notice that when the monetary policy shock is very persistent, the nominal interest rate can actually decline in response to a monetary contraction. This is because the downward adjustment of the nominal interest rate in response to the fall in inflation and the output gap more than offsets the effect of the initial shock. Nonetheless, the monetary policy shock is still contractionary since the real interest rate unambiguously rises.

Qualitatively, the dynamic responses to a monetary policy shock fit the data fairly well. However, matching some of the quantitative features requires several tweaks to the basic model. See Christiano et al. (2005) for further details.

4.5.6 Effects of a Technology Shock

Now consider the effects of a technology shock. Assume technology follows an AR(1) process (in logs), given by,

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a,$$

where $\{\varepsilon_t^a\}$ is a zero mean white noise process. Given (4.80), the natural real interest rate is given by

$$\hat{r}_t^n = -\sigma\psi_{ya}^n(1-\rho_a)a_t.$$

Assume that $\upsilon_t = 0$ for all t. Once again, we can use the method of undetermined coefficients to solve the model. Given that $\hat{r}_t^n$ enters the equilibrium system the same way as υ_t, only with opposite sign, the model solution is given by

$$\begin{aligned}\tilde{y}_t &= (1-\beta\rho_a)\Lambda_a\hat{r}_t^n \\ &= -\sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a a_t\end{aligned}$$

and

$$\begin{aligned}\pi_t &= \kappa\Lambda_a\hat{r}_t^n \\ &= -\sigma\psi_{ya}^n(1-\rho_a)\kappa\Lambda_a a_t,\end{aligned}$$

where

$$\Lambda_a \equiv \frac{1}{(1-\beta\rho_a)[\sigma(1-\rho_a)+\phi_y] + \kappa(\phi_\pi - \rho_a)} > 0.$$

When $\rho_a < 1$, a positive technology shock leads to persistent declines in the output gap and inflation. The responses of output and employment are

$$\begin{aligned}y_t &= y_t^n + \tilde{y}_t \\ &= \psi_{ya}^n(1-\sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a)a_t + \vartheta_y^n\end{aligned}$$

and

$$\begin{aligned}(1-\alpha)n_t &= y_t - a_t \\ &= [(\psi_{ya}^n - 1) - \sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a]a_t + \vartheta_y^n\end{aligned}$$

Thus, the sign of output and employment is ambiguous, depending on the particular calibration of the model. With log utility, positive technology shocks leads to persistent declines in labor.

4.5.7 Rotemberg Quadratic Price Adjustment Costs

Following Rotemberg (1982), assume each firm faces a quadratic cost to adjusting its nominal price level, which emphasizes the potentially negative effect that price changes can have on customer-firm relationships. Then, using (4.57) and adopting the functional form used in Ireland (1997), real profits of firm i are given by

$$\begin{aligned}\frac{D_t(i)}{P_t} &= \frac{P_t(i)}{P_t}Y_t(i) - \frac{\mathrm{TC}_t(i)}{P_t} - \mathrm{Adj.\,Cost}(i) \\ &= \left[\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon} - (1-\alpha)\,\mathrm{MC}_t\left(\frac{P_t(i)}{P_t}\right)^{-\tilde{\varepsilon}} - \frac{\varphi}{2}\left(\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right)^2\right]Y_t,\end{aligned}$$

where $\varphi \geq 0$ determines the magnitude of the adjustment cost and π is the steady-state gross inflation rate. Each intermediate goods producing firm then chooses their price level to maximize the expected discounted present value of real profits given by

$$E_t\sum_{s=t}^{\infty}Q_{t,s}\frac{D_s(i)}{P_s}.$$

The first order condition is given by

$$\begin{aligned}0 =\ &(1-\varepsilon)\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\frac{Y_t}{P_t} + \varepsilon\,\mathrm{MC}_t\left(\frac{P_t(i)}{P_t}\right)^{-(\tilde{\varepsilon}+1)}\frac{Y_t}{P_t} \\ -\ &\varphi\left(\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right)\frac{Y_t}{\pi P_{t-1}(i)} + \varphi E_t\left[Q_{t,t+1}\left(\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right)\frac{P_{t+1}(i)Y_{t+1}}{\pi P_t(i)^2}\right].\end{aligned} \tag{4.86}$$

Assuming a symmetric equilibrium, in which all intermediate goods producing firms make identical decisions, we can drop the i subscripts so that $Y_t(i) = Y_t$, $N_t(i) = N_t$, $P_t(i) = P_t$, and $D_t(i) = D_t$. Thus, after imposing these conditions and multiplying by $P_t(i)/Y_t$, the optimal pricing condition of the firm, (4.86), reduces to

$$\varphi\left(\frac{\Pi_t}{\Pi} - 1\right)\frac{\Pi_t}{\Pi} = (1-\varepsilon) + \varepsilon\,\mathrm{MC}_t + \beta\varphi E_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{\Pi_{t+1}}{\Pi} - 1\right)\frac{Y_{t+1}}{Y_t}\frac{\Pi_{t+1}}{\Pi}\right\}. \tag{4.87}$$

In the absence of costly price adjustment (i.e., $\varphi = 0$), real marginal costs reduce to $(\varepsilon - 1)/\varepsilon$, which is equivalent to the inverse of the firm's markup factor, $\mathcal{M}$.

Finally, the aggregate resource constraint is given by

$$C_t = \left[1 - \frac{\varphi}{2}\left(\frac{\Pi_t}{\Pi} - 1\right)^2\right]Y_t. \tag{4.88}$$

To summarize, equations (4.42), (4.43), (4.55), (4.56), (4.87), and (4.88) represent a system in W_t/P_t, C_t, N_t, I_t, Π_t, Y_t, and MC_t (7 variables) that can be combined with a specification of monetary policy to determine the economy's equilibrium (7 equations). To this system, we could also add exogenous processes that govern technology and monetary policy shocks as in previous sections.

The log-linear equilibrium system of equations is given by (4.67), (4.68), (4.69), (4.74), and

$$\hat{y}_t = a_t + (1-\alpha)\hat{n}_t, \tag{4.89}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{\varepsilon - 1}{\varphi}\widehat{\mathrm{mc}}_t. \tag{4.90}$$

Using (4.77), we can write the Phillips curve, given in (4.90), in terms of the output gap. Thus,

$$\pi_t = \beta E_t \pi_{t+1} + \frac{\varepsilon - 1}{\varphi}\left(\sigma + \frac{\eta + \alpha}{1-\alpha}\right)\tilde{y}_t, \tag{4.91}$$

which has the same functional form as the Phillips curve derived with the Calvo pricing mechanism, (4.78), but with a slightly modified Phillips curve slope. The output gap is given by (4.76) and the IS equation is given by (4.79). Thus, both the Calvo pricing and Rotemberg pricing mechanisms reduce to log-linear equilibrium systems composed of the Phillips curve, the IS equation, and a monetary policy rule (3 equations). However, it is important to note that with Calvo pricing, the model can be summarized by the three core equations only if the gross steady state inflation rate is one. Otherwise, the Calvo system must be expanded to include the x_1, x_2 and δ variables. Because of the simplification a zero inflation steady state offers, many of the early papers made this assumption.

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